GEOMETRICAL THINKING

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1. INTRODUCTION

The chapter provides a comprehensive view of the research in geometry education as occurred through CERME conferences, since the creation of the geometry group in CERME 3. For writing this chapter, the proceedings of all the CERME conferences since the formation of this working group were read again in order to be able to track the main developments in trends, theories and methodologies and the progress achieved in the research on the teaching and learning of geometry. Undeniably the research on the teaching and learning of geometry in the CERME conferences has developed through the creation of the group, emphasizing on the development of common theoretical frameworks and research trends regarding the teaching and learning of geometry in different educational levels and systems. This facilitated the attainment of communication and collaboration between researchers from different countries and levels of experience in the field of geometry education with researches in didactics of geometry.

In the first section, the evolution of topics which the group dealt with, are traced during the last nine CERMEs. The point is stressed on the close link between theoretical and empirical aspects which has always guided researchers of the group. Then, some methodological and theoretical tools developed during the activities of the group are introduced and we show how they gave a common point to support studies in the field. In the third section, main findings that have emerged from the group are given by insisting on the spirit of communication and collaboration built through the different meetings. In a concluding evaluative / projective section, a forward as well as a backward view in time leads us to envisage a common research agenda which could or should be organised to reach a better and deeper understanding of geometrical thinking and competencies through the whole education.

2. LINKING THEORETICAL AND EMPIRICAL ASPECTS: A VIEW ON THE HISTORY OF THE GROUP

The researchers of the group displayed a growing interested in using frameworks that allow the connection between theoretical and empirical aspects. In fact, cognitive and semiotic approaches were adopted such as the Van Hiele levels, Fischbein's figural concept and Duval's registers. The van Hiele model was continually used as a framework in several studies, whereas the reference to psychological approaches to geometry linked to spatial abilities (Gestalt and Piaget's theories) was present but not always explicit. The concept of tool mediation, central to the Vygotskian perspective, was sometimes present in background to support the analysis of the cognitive development. Regarding the epistemological and didactical approaches, a decisive importance was granted to the geometrical paradigms and the geometrical working spaces, and more recently to a lesser extent, on the notion of geometrical competencies.

A great number of research focused on different aspects of geometrical understanding, such as figural apprehension, visualization and the effort of conceptualization in geometry using different methodologies and frameworks. Actually, the researchers of the group have worked systematically on visualization processes and the different registers involved in the apprehension of geometrical figures, the discursive-graphic reasoning or during the use of technological devices, just as the interpretation of the student activity using developmental stages and categories when doing geometry. Among the numerous papers presented in the working group on Geometry, we can distinguish some recurrent topics that we present below.

Birth of the thematic group

The first working group specifically dedicated to geometry was only created in CERME 3. Since this meeting various names were given to this group including some nuances, from *thinking* to *research* through *teaching* and *learning*, ending at the last edition (CERME 10) with the sole name *Geometry* (Table 1). Where was hidden geometry in the first two conferences, or better yet, what kinds of studies were concerned with geometry?

Table 1a and b
Listing of the groups of geometry since CERME 3

Group	Title
Thematic Group 7 CERME 3	Geometrical Thinking
Working Group 7 CERME 4	Research on Geometrical Thinking
Working Group 7 CERME 5	Geometrical Thinking
Working Group 5 CERME 6	Geometrical Thinking

Group	Title
Working Group 4 CERME 7	Geometry Teaching and Learning
Working Group 4 CERME 8	Geometrical Thinking
Thematic Working Group 4 CERME 9	Geometrical Thinking
Thematic Working Group 4 CERME10	Geometry

In the first two sessions, even if the group did not exist, papers were presented and centred on geometry as a generic content, like other traditional domains such as algebra or statistics. The papers intend to provide references concerning curriculum issues (intended and implemented mathematics curricula in Europe) and the reform of mathematics examination at a time when social and professional mobility is a major issue (CERME 1: Kilpatrick). Contributions include also epistemological and cultural approaches about geometrical knowledge (CERME 1: Arzarello, Burton, etc., or the need of metaphors and images (CERME 1: Parzysz).

These studies are mostly descriptive, they address the geometric content indicating some common concepts or processes in school, sometimes even in comparison with the place of algebra in the curricula or textbooks. We often mention some typical interactions in the use of computing tools as the interactive geometry, "frictions" with the other mathematical domains, and links with general mathematical skills or the cognitive and instrumental dimensions of the teaching and learning of mathematics. They often mention some links with the geometry content and the general mathematical skills, the cognitive and instrumental dimensions in the geometric activity, the intramathematical modelling with geometry or some "frictions" with the other mathematical domains, and some typical interactions in the use of technologies as the interactive geometry.

In a more specific way, papers envisage algebra and linear algebra in their relations to geometry (for example, as linear algebra through geometry), especially to highlight the heuristic or visual aspects of these relations using interactive geometry software, and the dual status of the figures in the geometric transformations (structure or process, image or function, etc.) (CERME 1: Dreyfus, Hillel & Sierpinska; CERME 2: Stehlíková & Jirotková; Cerulli; Chartier; Rogers). Otherwise, the natural relationship of the geometry with proofs and proving is regularly examined, both in theoretical and philosophical essay and in reports of studies involving empirical and or developmental research (CERME 1: Douek; Grenier & Payan; Healy & Hoyles; CERME 2: Gallopin & Zuccheri; Mogetta). They face up the relationship between proving, arguing, modelling and discovering processes, from the student's performance with proofs to the dynamism of the discourse, the use of figural signs and the technology in the exercise of proofs in geometry. The intrinsic links between the discovery process, the visualization and the interactive geometry pay a special attention in several research (CERME 1: Zuccheri; Gutiérrez, Laborde, Noss & Rakov; Hoyos, Capponi & Geneves; CERME 2: Jones, Lagrange & Lemut; Olivero & Robutti). Upstream, there are studies that concern the teacher training in geometry (CERME 2: Kuzniak & Houdement) and the organization of courses, in particular with the use of interactive geometry and the symbolic computation software (CERME 1: Rakov & Gorokh), and downstream, other research presents results about student interpretations with the use of technology (CERME 1: Jones). Finally, some papers are more finely interested in the semiogenesis for the construction of figures (CERME 2: Maracci), in the phenomenon of conceptualization in geometry (CERME 2: Meissner) and in the mediation of the discourse to understand figural signs in their relations with a theoretical framework of geometrical definitions and properties (CERME 2: Robotti). The notion of procepts in geometry, in its meaning of an encapsulated process in a concept, is put handed to the front scene.

If mathematical proofs, visualization and interactive geometry appear in several works, it is surprising that other natural links with the geometry are virtually absent in the first congresses. We can mention that few research focuses on modelling of physical phenomena using geometric culture, or no paper deals with solving problems in geometry that are not problems of proof, or that do not focus only on the discovery of some characteristic properties well defined and known in advance.

Sustainable development for the research in geometrical education

If the simultaneous creation of autonomous working groups on *argumentation and proof*, *applications and modelling*, *language and mathematics*, and *technologies and resources in mathematics education*, to mention only the most apparent, has certainly had an effect on the orientation of the contributions, the following themes recur regularly from the beginning of the group on geometry: Development of spatial abilities and geometrical thinking through consecutive educational levels; Understanding and use of geometrical figures and diagrams; Geometry education and the real world: geometrisation and applications; Instrumentation: use of artefacts (e.g. mirror, computers, origami, prototype development, mathematical machines); Explanation, argumentation and proof in geometry education; Teaching of geometry and teachers' training. In spite of this opposition of tendencies (estrangement of contributions towards other groups versus convergence of themes in the group of geometry), certain approaches from collaborators of the group manage to connect the theoretical and empirical aspects in a prospect stemming from the domain of the geometry, as *geometrical paradigms* (CERME 3), *geometric working space* (CERME 5), and *geometric competencies* (CERME 8), enriching and pursuing the achievements of Van Hiele, Fischbein and Duval.

Table 2
Main topics in the CERME 6

Theoretical and methodological aspects of research in geometry
Educational goals and curriculum in geometry
Understanding and use of geometrical figures and diagrams
Understanding and use of concepts and "proof" in geometry.
Communication and assessment in geometry
3D Geometry: Teaching, thinking and learning

The organization of the work within the groups follow three styles of development, by summarizing the contribution of the group starting from recurring topics (table 2), by determining the theme or issues generated from papers (Figure 3) or, by according to some unifying characteristics, as geometric competencies (Figure 4). If certain questions remain transverse, others on the contrary are very close to the geometrical education. The originality of the group of geometry is as much in the means used to organize the work (convergent and mutual enrichment of the contributions) as in the emergence of promising results and the pursuit of the work from an edition to another.

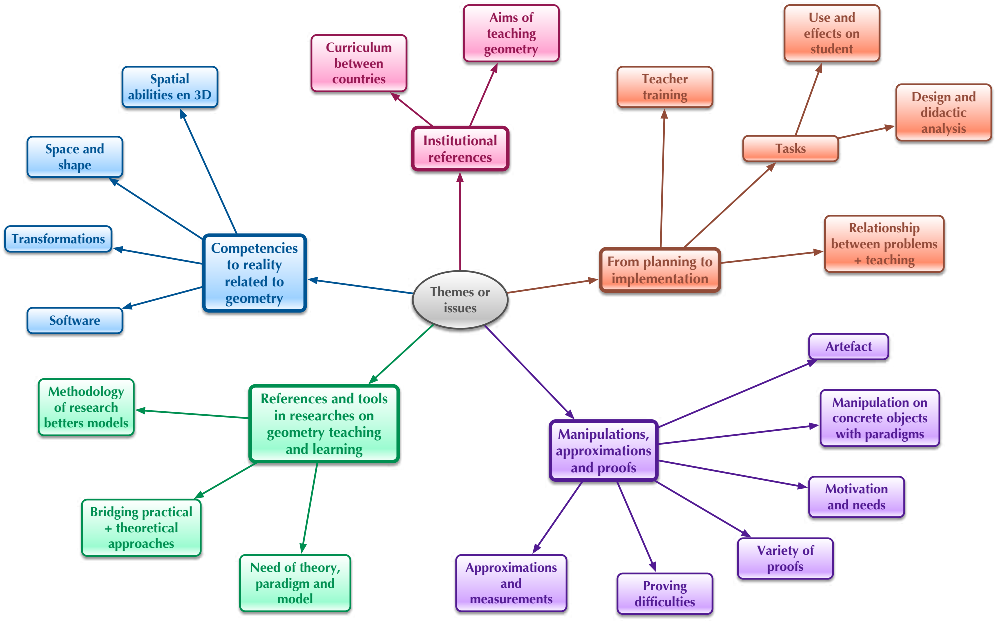

Figure 3. Theme or issues presented in the CERME 7

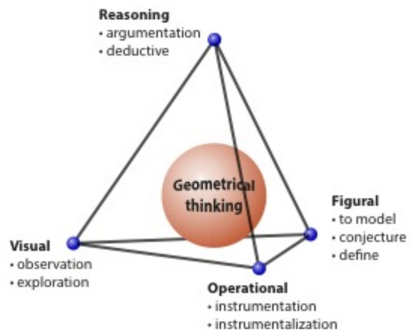

Figure 4. The geometry competencies in the CERME 8

Thematic from CERME 3 to CERME 5

Following the logic of the dominant themes, while simplifying terribly for the purposes of this chapter, the group on geometry start to study the geometrical thinking with the key concepts of *paradigms*, *developmental stages* and *generalization in space* (CERME 3: Houdement & Kuzniak; Gueudet-Chartier; CERME 5: Houdement). If *visualization* is in the core of much research, it is often directly connected with the *registers of representation* (CERME 3: Vighi; Acuña; Lanciano; Perrin-Glorian; Kurina; Bakó; CERME 5: Panaoura, Gagatsis & Lemonides; Pittalis, Mousoulides & Christou), sometimes in spatial geometry (CERME 3: Cohen), and using the notion of *instrumentation*, with a predominance for interactive geometry (CERME 3: Larios-Osorio; Rolet; CERME 4: Vighi; CERME 5: Bulf; Kospentaris & Spyrou; Larios-Osorio; Mann & Ludwig; Vighi). Even in an original study, the issue is raised by sighted and non-sighted pupils (CERME 5: Kohanová). The *teach future teachers* (CERME 4: Kuzniak & Rauscher) and the *reasoning* (CERME 4: Ding, Fujita & Jones; Markopoulos & Potari) appears as topics of continuity, while *concepts and conceptions* remain a key theme in many studies (CERME 3: Kónya; Prokopová; CERME 4: Marchini & Rinaldi; Marchett, Medici, Vighi & Zaccomer; CERME 5: Krátká; Marchini & Vighi; Markopoulos, Potari & Schini; Modestou, Elia, Gagatsis & Spanoudes).

Thematic from CERME 6 to CERME 7

In these editions, the theoretical and methodological dimensions of research in geometry have been treated as a prominent topic, especially in pursuing the notions of geometrical work and the geometrical working space. In addition to the usual topics (geometrical figures, diagrams, proofs or 3D Geometry), a special attention was paid in more general or transversal aspects as educational goals and curriculum, communication and assessment, the teaching, the thinking and the learning processes in geometry. We propose new studies on the *geometric transformations* (CERME 6: Bulf; Kali & Panagiotis; CERME 7: Marchini & Vighi; Jagoda & Swoboda; Xistouri & Pitta-Pantazi), *from the geometric transformations to the teacher training* (CERME 7: Thaqi, Giménez & Rosich; Fonseca & Cunha), about *spatial abilities*, the *figure reasoning* and the *modelling* (CERME 6: Girnat; Pandora & Gagatsis; Deliyianni, Elia, Gagatsis et al.; Kunimune, Fujita & Jones; Semana & Santos; Deliyianni, Gagatsis, Monoyiou et al., Mithalal; Pittalis, Mousoulides & Christou; González & Guillén; Vighi; Acuña; Levav-Waynberg & Roza Leikin; CERME 7: Gagatsis, Michael, Deliyianni et al.; Braconne-Michoux), *on curriculum and general geometrical work* (CERME 6: Kuzniak & Vivier; CERME 7: Girnat; Kuzniak; Bulf, Mathé & Mithalal), linking *reasoning and technology* (CERME 7: Fujita, Jones, Kunimune et al.; Steinwandel & Ludwig), and *on the dynamic environment* and the *interactive interactive geometry software* (CERME 6: Hattermann; CERME 7: Forsythe; Mackrell).

Restructuring in the recent CERME 8 and CERME 9

If the raised themes are appreciably the same as from the previous editions, the working group used the idea of four geometrical competencies as unifying characteristics (Fig. 4). Each competency acts as a pole of interest around geometrical thinking: when a contribution focuses on the issues or the effects of a given competency, for example, the argumentation and deduction in the reasoning competency, the links generated with the plane of the other three competencies contribute to a better understanding of the geometrical thinking. The question of the coordination between competencies has brought closer the theoretical and empirical aspects of the approaches implemented in the studies, facilitating the comparison of the stakes.

3. STUDYING THE TEACHING AND LEARNING OF GEOMETRY THROUGH A COMMON LENS

From the beginning, the development of shared theoretical frameworks was central to ground collaboration between participants of the group. This point is particularly clear in Cerme3, where the main theoretical concerns of the group were listed (…) and we use it to highlight the theoretical approaches which have been reinforced by their presentation and discussion during Cerme conferences.

Geometrical Paradigms

The history of geometry shows two contradictory trends. First, gometry is used as a tool to deal with situations in real and economic life but, on the other hand, geometry for more than two thousand years was considered the prototype of logical, mathematical thinking and writing after the publication of Euclid's "Elements". These contradictory perspectives are somehow mirrored in an approach widely discussed from Cerme3 in the Topic Group, when Houdement and Kuzniak presented their approach based on geometrical paradigms to take into account the diversity of points of view on geometry education. They distinguish "Geometry I: natural geometry" (source of validation: sensitive; this geometry is intimately related to reality; experiment and deduction act on material objects), from "Geometry II: natural/axiomatic geometry" (hypothetical deductive laws are the source of validation; the set of axioms is as close as possible to intuition and may be incomplete; certainty is secured by demonstrations) and "Geometry III: formal axiomatic geometry" (axioms are no more based on the sensory reality, should be complete in the formal sense and independent of applications; consistency is the criterion of existence). These various paradigms are not organized in a hierarchy making one more preferable than another, but their work horizons are different and the choice of a path towards the solution is determined by the purpose of the problem and researcher's viewpoint.

The discussions showed that this more detailed description of geometry provides a method to classify geometrical thinking. It can also be helpful to interpret tasks eventually given to students and future teachers and can be used to classify the productions from the students, offering an orientation for the geometry teacher. In being closer to the learner's and teacher's world (at least before entrance to university) it also avoids the pitfalls of linear algebra / analytical geometry which often is so far off the geometry world of the student that it is no more taken as part of geometry (see also the paper by Gueudet-Chartier). Some question were asked on possibilities of lower secondary teachers to 'reach' an insight into geometry which includes the Geometry III level at all.

Development stages

The so-called 'Van Hiele levels' of geometrical reasoning are nowadays one of the most used theoretical framework to organise the teaching and learning of geometry according development stages. The "Principles and standards for school mathematics" (NCTM, 2000) are a clear example of application of Van Hiele levels to curriculum design.

The description of the 'van Hiele levels' already gives hints to fundamental links between these levels and the model suggested by Houdement and Kuzniak. Consequently and following the argumentation in the paper by Houdement and Kuzniak, these links were widely discussed in the Cerme3. There was some discussion if geometrical knowledge progresses through sequences of stages. Some of the papers in the group could be seen as contradictory to the traditional 'van Hiele levels'. This is especially true if the levels are linked to clearly identified and fixed ages or if individual persons are thought to necessarily follow the order of the respective levels. A position on these ideas on developmental and/or learning stages is obviously important when geometrical tasks or research projects are being constructed or discussed. If accepted, the levels can help to find and further develop appropriate tasks for developmental and research work – and they are obviously helpful for explorative activities to come across new, maybe even innovative ideas.

Registers of representation

Semiotic consideration has always been important in geometry and the distinction figure, in the sense of the most general object of geometry (either 2D or 3D), vs its material representation is classic in the field. Three main groups of semiotic representation can be distinguished: material representation (in paper, cardboard, wood, plaster, etc.), a drawing (made either with pencils on a sheet of paper, or on a computer screen, with use of a geometric software, etc.), and a discursive representation (a description with words using a mixture of natural and formal languages). Each register bears its own internal functioning, with rules more or less explicit. Moreover, students have to move from one register to another, sometimes explicitly, sometimes implicitly, sometimes back and forth. Questions about registers of semiotic representation and cognitive processes have been studied in depth by Duval (1995). He defines "semiotic representations as productions made by use of signs belonging to a system of representation which has its own constraints of meaning and functioning". Semiotic representations are absolutely necessary to mathematical activity, because its objects cannot be directly perceived and must, therefore, be represented. From the beginning, various papers has been based on this theoretical approach and we will come back on it in the following.

Intrumentation with artefacts, computers, etc.

There is broad consensus in the community of mathematics teachers and educators that learning geometry is much more effective if concepts, properties, relationships, etc. are presented to students materialised by means of instruments modelling their characteristics and properties. Furthermore, the use of didactic instruments is very convenient, if not necessary, in primary and lower secondary grades. There is a huge pile of literature reporting the continuous efforts devoted by mathematics educators since long ago to explore the teaching and learning of geometry with the help of manipulatives, computers, and other tool.

On Geometrical Working Spaces

Related to semiotic, instrumental and discursive proof, the idea of thinking the geometric work globally was first introduced in Cerme 5 (Kuzniak and Rauscher) and was based on the model of GWS. The basic idea behind this approach is that some real geometrical work appears only when student's activity is both coherent and complex to develop a reasoning using intuition, experimentation and deduction. The GWS is conceived as a dynamic abstract place that is organized to foster the work of people solving geometry problems. The model articulates two main planes, the one of epistemological nature, in a close relationship with mathematical content of the studied area, and the other of cognitive nature, related to the thinking of the person solving mathematical tasks.

This complex organization is generally summarized thanks to the two following diagrams (Fig. 5 and Fig. 6) (For details Kuzniak and Richard 2014, Kuzniak, Tanguay and Elia 2016):

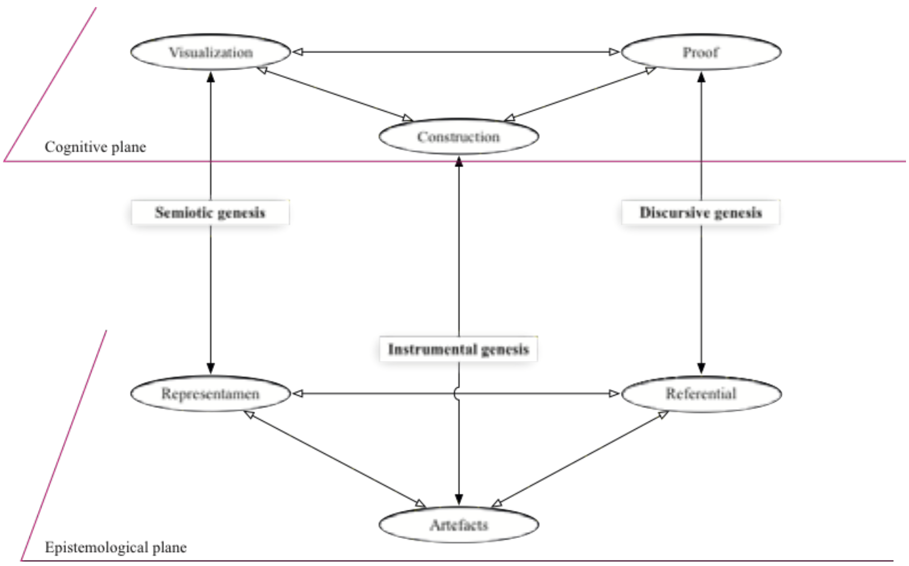

Figure 5. The Mathematical Working Space Diagram

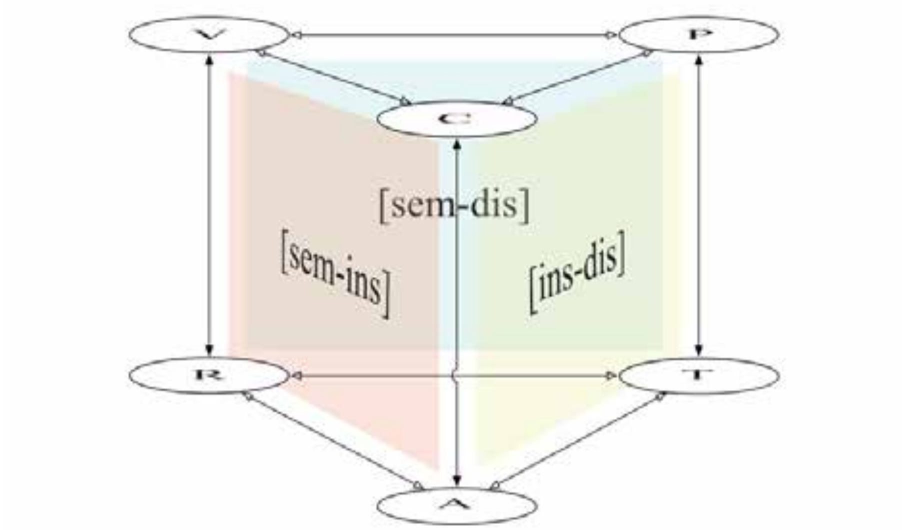

Figure 6. The three vertical planes in the MWS

Three components in interaction are characterized for the purpose of describing the work in its epistemological dimension, organized according to purely mathematical criteria: a set of concrete and tangible objects, the term sign or representamen is used to summarize this component; a set of artifacts such as drawing instruments or software; a theoretical system of reference based on definitions, properties and theorems.

The second level of the GWS model is centered on the subject, considered as a cognitive subject. In close relation to the components of the epistemological level, three cognitive components are introduced as follows: visualization related to deciphering and interpreting signs; construction depending on the used artifacts and the associated techniques; proving conveyed through processes producing validations, and based on the theoretical frame of reference.

The process of bridging the epistemological plane and the cognitive plane can be identified through the lens of GWSs as genesis related to a specific dimension in the model: semiotic, instrumental and discursive geneses. This set of relationships can be described proceeding from the elements of the first diagram (Figure 5) which, in addition, shows the interactions between the two levels with three different dimensions or geneses: semiotic, instrumental, and discursive. The epistemological and cognitive planes structure the GWS into two levels and help to understand the circulation of the knowledge within mathematical work. How then, proceeding from here, students can articulate the epistemological and cognitive levels in order to do the expected geometrical work? In order to understand this complex process of interrelationships, the three vertical planes of the diagram are useful and can be identified by the genesis which they implement [Sem-Ins], [Ins-Dis] and [Sem-Dis] (Fig. 6). The precise study of the nature and dynamics of these planes during the solving of mathematical problems remains a central concern for a deeper understanding of the GWS model.

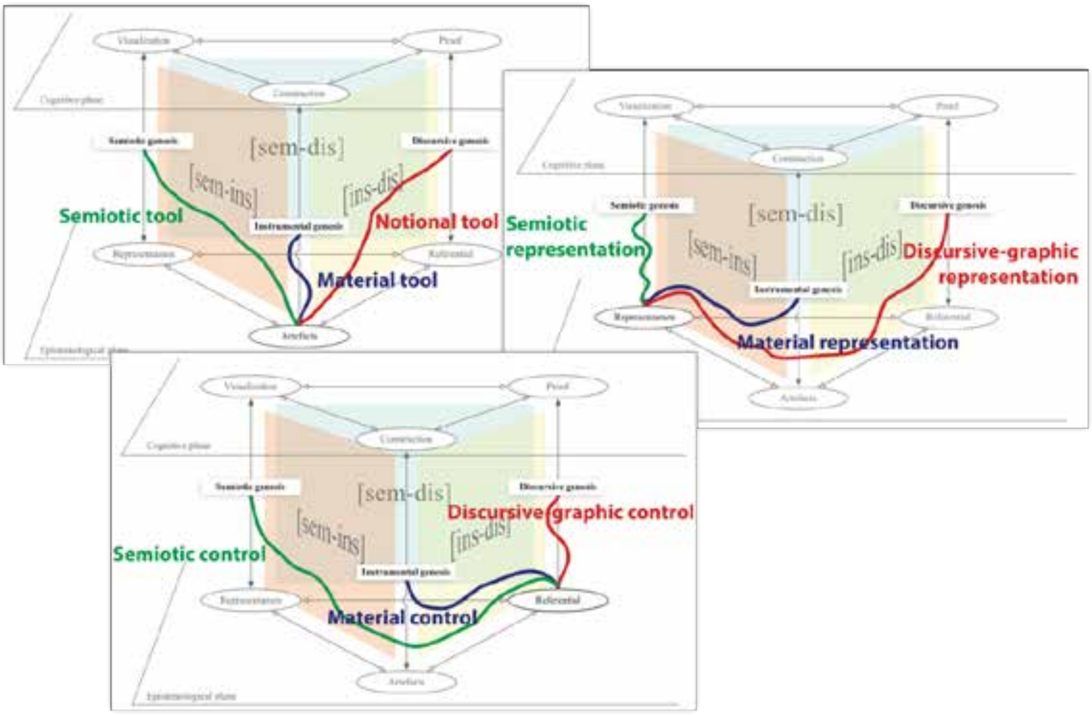

Figure 7. Three fibrations in the MWS

To the coordination of the geneses in the MWS model, there are three internal fibrations that focus on the role of tools, controls and representations in the mathematical work (Richard, Marcén & Meavilla, 2016). The first type of fibration (Fig. 7, top left), inspired by Kuzniak and Drouhard (2015), shows that semiotic, material and notional tools (notion in discursive instance that allows processing) have an effect on the geneses, since the use of these tools influences the formation of semiotic, notional and material parts of the instrument. The second fibration (Fig. 7, top right), which underpins the referential, indicates that the semiotic, material and discursive-graphic controls affect geneses, for instance, when controlling the sign, the software tool or the discourse of proof. The third type of fibration (Fig. 7, bottom left) from the representamen, to report the effect of the semiotic, material and discursive-graphic representation about the geneses, which makes possible to account the difference between an object and its representation in geometry, especially when we consider that the geometric representations: a) are also forms (shapes), and, as object-forms, they are both model and reality; b) act as objects-situations, especially in contexts instrumented by technological tools (Coutat, Laborde & Richard, 2016).

Using the GWS framework makes possible to question in a didactic and scientific – mostly non-ideological – way the teaching and learning of geometry.

4. BUILDING AND DEVELOPING A SPIRIT OF COMMUNICATION AND COLLABORATION

As previously underlined, a strong point of the group is participants' emphasis on relating theoretical and empirical aspects of research in geometry education. The continuity of this trend was evident through the meetings of the group so far, as the participants were mainly discussing results of empirical or developmental research studies and theoretical reports about the teaching and learning of geometry. The need for a common framework related to Geometry education appeared necessary in the working group in order to stimulate the discussions among members and to allow a capitalization of knowledge in the domain.

Due to collaborations initiated during CERME meetings with colleagues from France, Cyprus, Spain, Canada, Mexico and Chile, it has been made possible to develop a joint theoretical framework. The framework should be dedicated to study the teaching and learning of geometry, space and shape on the whole educational system and should be neutral in the sense that it can be used to facilitate exchanges in different countries and institutions. According to this need, the framework of geometrical paradigms, as explained above, was introduced in the CERME 3 conference. In the pursuit of the results at the CERME 3, the geometry working group of CERME 4 and 5 continued by looking into geometrical paradigms.

In particular, in CERME 4, Kuzniak and Rauscher (2005) analyzed pre-service schoolteachers' geometrical approaches, based on the notion of geometrical paradigms and levels of argumentation. They found that students' levels of understanding and memorization of the bases of the elementary geometry differ greatly and that they keep the practical use of Geometry. Their results revealed also two approaches used by the students; the first one based only on visual indications while the second uses results obtained thanks to the instruments of construction and measuring. Although their study was conducted with a particular population, their results can be useful for evaluating the long-term effects of education in geometry.

Moving forward, in CERME 5, during discussing the possible uses of geometrical paradigms, new participants of the group initiated a discussion about the real benefit of this approach. Perspectives in this directions were given by Houdement's (2007) and Bulf's (2007) papers. In fact, Houdement highlighted the uses of this approach for comparing curricula in different countries and for reflecting on the necessity to teach Geometry II and the proper way to introduce it. In Bulf's effort to examine the link between geometrical knowledge and the reality in relation to the concept of symmetry, this approach was useful for tracing a double play between the Geometry I and Geometry II on one side and Reality/Theory on the other side. Furthermore, Kospentaris and Spyrou (2007) used the approach of paradigms in teachers' training, as done in CERME 4 by Kuzniak and Rauscher (2005). Their results were in line with previous results presented at CERME 3 and 4 about pre-service teachers' geometrical thinking. They actually found that visual strategies or measurement using tools are used by students at the end of secondary school, interpreting it not by a developmental approach, but based on the geometrical paradigms. The discussion of the aforementioned papers gave a future perspective for the group in order to precise the sufficiency of the so far existing theoretical tools for determining the nature and the construction of the Geometrical Working Space used by students and teachers.

Following up the spirit of the previous years, the participants in CERME 6, who come from both Europe and America, have extended and enriched the results obtained so far. Until then a common background was built and known by experienced participants, thus the participants worked within the continuity of the former sessions of CERME and their discussions were effectively facilitated by this common culture. The theory of Geometrical Working Space and Geometrical paradigms was among the most important theories in geometry education that were used by the participants. In fact, the participants came to two main conclusions regarding the use of theory in research: (1) theory can serve as a starting point for initiating a research study and (2) theory can act as a lens to look into the data. An example of such research is the one of Kuzniak and Vivier (2009) who examined the Greek Geometrical Work at secondary level from the French viewpoint, using a theoretical frame based on paradigms and geometrical working spaces. Also, Panaoura and Gagatsis (2009) compared the geometrical reasoning of primary and secondary school students, based on the way students confronted and solved specific geometrical tasks, finding difficulties and phenomena related to the transition from Natural Geometry to Natural Axiomatic Geometry. Therefore, a perspective for future research on geometry theories and their articulation for the group was the use of Geometrical Paradigms as a tool for analyzing existing curricula and students' behavior.

In the subsequent CERME conferences, this common culture was not only sustained; attendees gave a better understanding on several questions that addressed the issues covered in the previous works. Actually, in CERME 7, many points were considered as a common background, as they were developed during former sessions. These points were related to educational goals and curriculum in geometry (Girnat, 2011; Kuzniak, 2011), to the use of geometrical figures and diagrams (Deliyianni, Gagatsis, Monoyiou, Michael, Kalogirou & Kuzniak, 2011) and to the understanding and use of concepts and proof in geometry (Gagatsis, Michael, Deliyianni, Monoyiou & Kuzniak, 2011; Fujita, Jones, Kunimune, Kumakura & Matsumoto, 2011). For an epistemological and didactical approach, researchers used the geometrical paradigms and geometrical work spaces.

More recently, the working group at CERME 8 sought to revisit and extend the issue of geometrical work by reformulating it in terms of four geometrical competencies (reasoning, figural, operational and visual). The group has set each geometrical competency as a pole of interest around geometrical thinking, in a way that if one focuses on the issues, or the effects, of a given competency (eg, argumentation and deduction in the reasoning competency), links are generated with the plane of the other three competencies (eg, visual-figural-operational) that also contribute to a better understanding of the competency in question (Maschietto, Mithalal, Richard & Swoboda, 2013). Further on, in the working group at CERME 9, there were more specific contributions about the way geometry is, or should be, taught and the four competencies were used as a general way of describing the geometrical activity and for creating links between different points of view (theoretical and empirical). In this group the discussions were related to geometry teaching and learning (Douaire & Emprin, 2015; Kuzniak & Nechache, 2015) and issues like teaching practices and task design (Mithalal, 2015; Pytlak, 2015). Furthermore, cultural and educational contexts modifying the geometry curricula were also discussed, introducing a new issue about the role of language and social interactions in the teaching and learning of geometry.

The creation of a common spirit of communication has also built ideas of collaboration between participants through the discussions of the group. This was evident regarding the focus of research in specific educational levels. Actually, at the first discussions of the working group the attention was given on primary education, as many of the papers were about young students' geometrical concepts (Marchetti, Medici, Vighi & Zaccomer, 2005 ; Marchini & Rinaldi, 2005) and the role of specific tools for the teaching and learning of Geometry at that age level (Vighi, 2005).

However, in next meetings of the group (in CERME 5), collaborations were envisaged about the transition from a lower to a higher educational level and also the adaptation of a common framework to work out such kinds of studies, like paradigms, geometrical working space, spatial abilities and conceptions about the figure.

This was succeeded in the next meeting of the group in CERME 6, as among the research presented in the group the dimension of the students' transition from primary to secondary school was also taken into account. For example, Deliyianni et al. (2011) investigated the role of various aspects of figural apprehension in geometrical reasoning in relation to the students' transition from primary to secondary education, revealing differences between the two groups of students' performance and strategies in solving geometrical tasks. In a similar sense, Panaoura and Gagatsis (2013) compared primary and secondary school students' solutions of geometrical tasks and stressed the need for helping students progressively move from the geometry of observation to the geometry of deduction as they transit to a higher educational level.

Finally, during the effort of building a spirit of communication and collaboration in the group, collaborations between experienced and new researchers were accomplished, while using the common framework of geometrical paradigms and workspace (e.g the work of Deliyianni et al., 2011 and Gagatsis et al., 2011). These common works facilitated not only the communication between old and new researches, but also the collaboration between researchers from different countries (e.g France and Cyprus).

5. A Common Research Agenda: Understanding geometrical work and competencies through the whole education.

Even if exchange within the group is still very rich and exciting, the geometry group seems to have gradually forsaken some of its initial ambitions, because of the existence of various groups specifically interested in technology use, proof, teacher training, semiotic aspects... It has been partially disembodied and deprived of what has always been the strength of geometry: its transdisciplinary contribution to human thinking.

Another challenge faced by the group comes from its difficulty to capitalize its results and findings because of two kinds of volatility. The first is natural and comes from the renewal of participants who are younger and younger, sometimes, beginners in the field. Experimented researchers were attracted by other groups or by other more attractive form of working groups in other contexts.

Another reason of this difficulty is the constant curriculum changes in geometry. This can be illustrated by the erratic presence of geometric transformations. Moreover, and fundamentally, geometrical activity seems more and more oriented to other mathematical fields through modelling activity supported on geometrical support, such as physics, geography or film studies.

For the future and to overcome some of this trouble, we suggest that the viewpoint on geometrical work could help to shape a common research agenda aiming at understanding better competencies involved in geometrical work though the whole education. It requires coordination between cognitive, epistemological and sensible approaches, structured around three complementary dimensions (semiotics, instrumental and discursive).

Understanding the semiotic work.

Geometry is traditionally viewed as a work on geometrical configurations which are both tangible signs and abstract mathematical objects. Parzysz (1988) has clearly identified this difference under the opposition *drawing vs figure* which focuses on strong interactions existing between semiotic and discursive dimensions. The semiotic dimension, especially worked through the visualization process, is at the centre of Duval's research who developed very powerful tools, such as registers of semiotic representation, to explore the question. In his view, a real understanding of mathematical objects requires an interplay between different registers which are the sole tangible and visible representations of the mathematical objects. In a semiotic perspective too, Arzarello (2006) used the notion of semiotic bundles to study the entanglement of mathematical objects and their various semiotic representations. One of the main issues will be to see how this link is formed and reformed and can hide the very nature of geometrical objects to students.

Understanding the instrumental work

Geometry could not exist without drawing tools and study of their different use allows identifying two types of geometry which are well described by GI and GII paradigms. From precise drawings but mathematically wrong constructions (like Dürer pentagon) to exact but imprecise constructions (like Euclid's pentagon), it is possible to see all the epistemic conflicts opposing constructions based on an approximation to constructions based on purely deductive arguments. The tension between precise and exact constructions has been renewed with the appearance of dynamic geometry software (DGS). As Straesser (2002) suggested, we need to think more about the nature of geometry embedded in tools, and reconsider the traditional opposition between practical and theoretical aspects of geometry.

Understanding the work around proof and reasoning.

Since antiquity, proof work in its demonstrative form has always been highlighted by and supported on geometry, viewed as a kind of ideal of rational thought seized in its most intuitive and visible form. But in education, this idealized form of advanced mathematical work does not appear so obvious hidden behind the play between practical and formal geometrical paradigms. Software stretches boundaries of graphic precision, and finally, ends to convince users of validity of results. Proof work does not remain simply formal and forms of argumentation are enriched by experiments which give new meaning to the classic epistemological distinctions between iconic and not iconic reasoning related to a discursive-graphic reasoning (Richard 2004a, b), and between a pure human reasoning to an instrumented reasoning that voluntarily delegates a part of the representation and processing (treatment) to a machine (Richard, Marcén & Meavilla, 2016). If we already know that reasoning controls the representation (semiotic aspect), we now understand better how it animates the figures (instrumental aspect) and how it is expressed by coordinating the figurative signs with those of the natural language (discursive aspect).

Understanding the whole work to complete it.

How all the complementary dimensions — semiotic, instrumental, discursive — relate to each other and, specifically, how new instruments use interacts with semiotic and discursive geneses in transforming discovery and validation methods? And how student's geometric work can be structured in a way which is not simply that of a without thought drudge? The notion of complete mathematical work defined by Kuzniak, Nechache and Drouhard (2016) can be helpful to develop resarches and tasks in geometry education.

a) A genuine relationship between epistemological and cognitive dimensions. This aspect means that students are they generic or not are able to select the useful tools to deal with a problem and then to use them appropriately as instruments to solve the given task.

b) An articulation of a rich diversity between the different dimensions of the work related to tools, techniques and properties are taken into account. This point reflects how different working contexts are involved during students' activity

Various theoretical tools presented during Cerme conferences can be used in analyzing, describing and interpreting how geometrical tasks can be relevant to reach the desired and complete geometrical work. Moreover, it helps teachers to develop and plan some series of tasks and problems for developing students' skills, and more particularly, geometrical competencies.

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