Number sense and teaching and learning of arithmetic

Sebastian Rezat, Lisser Rye Ejersbo

Introduction

Starting in 2011, a separate working group has been devoted to research on teaching and learning arithmetic and number systems at CERME. While the focus of the group encompassed grades 1 to 12 the majority of contributions related to the elementary and lower secondary grades. The TWG covered a range of issues related to the teaching and learning of numbers including number relations such as equality and proportion as well as related argumentation and reasoning. A dominant theme has been the conceptual understanding of natural and rational numbers and related representations mostly combined with the theme of flexibility and adaptiveness in mental calculation. Research in this field often refers to the concept of 'number sense', which seems to be both, the starting point and the goal of the development of a conceptual understanding of numbers.
Already in 2011 at the introduction of the TWG2 "Arithmetic and number systems" there were several contributions referring to the concept 'number sense'. The number of contributions has even increased and at CERME 8 and 9 more than half of the papers in TWG2 „Arithmetic and number systems" referred to number sense.
In fact, the concept number sense has been a theme at CERME conferences from the beginning also in other TWGs. However, the concept appears to be vague in the papers. A closer look reveals different meanings of the concept "number sense" and sometimes the very meaning is not even explicated. In many papers the development of number sense is mentioned as a goal without clarifying what it means to develop or possess number sense. In some papers it is implicitly defined by the research that is supposed to contribute to the development of number sense.
In this article we attempt to unveil the different notions of number sense that are used at CERME and complement these by definitions from other perspectives, especially from neuro-psychology. We will address four main questions:

1) What is the understanding of number sense within CERME?
2) How do the different understandings of number sense relate to one another?
3) What is suggested in order to foster the development of number sense?
4) What is the scope of number sense in terms of number domains?

Our main aim is to clarify different perspectives on abilities related to numbers that are subsumed under the term 'number sense' and to suggest a terminology, which differentiates between these different abilities. In order to identify papers related to number sense we searched for the term „number sense" in CERME-proceedings 1 – 9. This search revealed altogether 39 papers that contained the term number sense. We will summarize the main findings and discuss them in this paper.

Different understandings of number sense within CERME

A number of papers that refer to number sense do not explicate their understanding of the very term, but use it as an undefined term. In fact, most papers that characterize their understanding of number sense explicitly or implicitly relate to the definition provided by McIntosh, Reys, and Reys (1992, p. 3): “Number sense refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity“. (Hedrén 1999, Chrysostomou, Tsingi, Cleanthous, Pitta-Pantazi 2011, Meissner 2011, Carvalho & da Ponte 2013, Morais & Serrazina 2013). In this definition number sense is characterized as a person’s understanding combined with an ability and inclination to make use of this understanding. In that it shows similarities to the definition of competencies according to Weinert (2001). Although it is not said explicitly in McIntosh et al.’s definition, the close relation to competencies is likely to evoke the idea, that number sense can be developed through teaching. Therefore, we call the understanding of number sense according to McIntosh et al. ‘didactical understanding of number sense’, as opposed to an innate sense.

Some papers acknowledge that there are different understandings of number sense and mention both the innate and the didactical understanding of number sense. However, in most of the cases this is done in order to contextualize the study within one of the understandings, which is almost exclusively the didactical understanding.

The following five examples all mention that there are two different understandings; still, they are very different examples and give an idea of the variety of understandings of the very term “number sense” in CERME. There is no special line in the five examples other than they are examples of how authors use the concept number sense in many different ways. The first three examples are from TWG 2, while the two last ones are from other TWGs.

Example 1: Rechtsteiner-Merz and Rathgeb-Schnierer (2015).

In their article, Rechtsteiner-Merz and Rathgeb-Schnierer (2015) describe how to develop flexible mental calculation and relate it to problem characteristics, number patterns and numerical relationships. In line with that they introduce the concept ‘Zahlenblick’, which refers to both number sense and structure sense. As for number sense they write: “The term number sense is connected with two different notions: as a result of experience based development or as an inherent skill.”

As for the structure sense, they also mention two meanings: An early structure sense that is similar to the cognitive inherent skill and a more developed competence. ‘Zahlenblick’ is also a product of development: “Comparing number sense, structure sense and ‘Zahlenblick’ it is obvious that the meaning of number and structure sense is quite similar to our notion of ‘Zahlenblick’.” ‘Zahlenblick’ is to be developed through ‘Zahlenblickschulung’, which is a program that focuses on developing flexible mental calculation. The program can be understood as an essential principle of teaching arithmetic and is in that way a suggestion to a theoretical framework. Even though they do not mention McIntosh et al. (1992, p. 3), it is still this understanding they work into.

Example 2: Morais and Serrazina (2013)

Morais and Serrazina (2013) refer to Sowder (1992), who associates number sense with intuition and a flexible creative way of problem solving, Dehaene (2001), who refers to

approximation and manipulation of numerical quantities and McIntosh, Reys and Reys (1992). Their conclusion is that "a good number sense implies a thorough and flexible understanding of numbers and their relationship, essential for the development of efficient and useful strategies" (p. 333).
They mention Dehaene (1997) and interpret his ontological view of number sense as being similar to the understandings of Sowder and McIntosh et al. But when Dehaene talks about approximation it relates to magnitude and comparing magnitudes; in other words he refers to the question which group or which number is bigger and applies the same to manipulation of numeral quantities. As opposed to McIntosh et al. Dehaene (1997) explains his definition of number sense from a biological view. In his research he found an area in the brain, the Intra Parietal Sulcus (IPS) in the Parietal lobe, which enables us to sense exactly the size of a number, to compare numbers, and to determine the larger amount between two sets. This area he calls number sense. We are born with this area/module and so are other biological species as well. Number sense in the understanding of Dehaene (1997) is an evolutionarily derived sense, and it is a part of our natural intuition. The innate conception of number sense is closely related to the approximation number system (ANS). Using that research as a basis for didactical research causes an ontology, in which we believe that children are able to differ between different magnitudes and use this skill in the development of arithmetic understanding. In fact this ontological view is not clear in the rest of Morais' and Serrazina's article. They focus on the development of skills related to subtraction with regrouping and not on approximation.

Example 3: Sayers and Andrews (2015).
Sayers and Andrews (2015) differentiate preverbal, foundational and applied number sense. According to the authors "preverbal number sense reflects those number insights innate to all humans and comprises and understanding of small quantities that allows for comparison" (p. 361), whereas applied number sense relates to the understanding of McIntosh et al. (1992) and "concerns those number competencies related to arithmetical flexibility that prepare learner for an adult world" (p. 361). While preverbal refers to the innate number sense and applied refers to the didactical understanding, the authors introduce a third notion of number sense, called "foundational number sense", which "comprises those understandings that precede applied, typically arise during the first year of school and require instruction" (p. 361). Sayers' and Andrews' (2015) definition of foundational number sense relates to applied number sense using the notion of "preceding". Although the authors do not say explicitly that preverbal also precede foundational number sense, their definition of the foundational number sense conveys a notion of three different stages of number sense.

Example 4: Aunio, Aubrey, Godfrey, Liu and Yuejuan (2007) TWG 'Mathematics education in multicultural settings'.
Aunio, Aubrey, Godfrey, Liu and Yuejuan (2007) take a different approach to number sense. The authors have a very general understanding of number sense. For them „number sense entails operating with quantities and number-word systems" (p. 1577). In their study their reference for measuring number sense is the "Utrecht Early Numeracy Test (UENT)" they investigate the influence of age, gender, culture on number sense (as measured by UENT). Their findings with children of about 5 years from three different countries indicate that number sense develops differently in the three countries. „UENT measures two different aspects of the number-sense construct with the relational skills being less influenced by

teaching and language difference and, hence culture than counting skills“ (p. 1583). They conclude that there are two different structures of numerical skills: general numerical skills and specific numerical skills with the latter being affected by teaching.
Although the authors do not relate their results to the cognitive or didactical understanding of number sense, it seems reasonable that the general, relational skills are related to the innate number sense while the specific, counting skills relate to the didactical number sense. Thus, their results challenge the view that the different understandings of number sense can be regarded as stages. The two notions rather seem to be two aspects of number sense that are intertwined and work together. However, the question of how the cognitive and didactical view of number sense interrelate and how the development of the didactical number sense can be supported by building on the innate number sense seem not have to been tackled so far.

Example 5: Turvill (2015) TWG ‘Diversity and Mathematics Education’.
In her article Turvill (2015) refers to two different systems for processing numerosity, offered from cognitive psychology. The two systems are subitizing and approximate number system (ANS), which both are taken to be innate and are contained in the cognitive concept of number sense. She refers to Xu & Spelke (2000), who suggest that number sense develops spontaneously and have demonstrated how children as young as six months can discriminate between sets of two and three. Halberda & Feigenson (2008) demonstrate how number sense is developed throughout the infancy and can be used in schools, while Feigenson, Libertus, & Halberda (2013) show how regularly use of number sense influences individual mathematics abilities because it is lifelong.
Turvill also refers to McIntosh et al.’s definition of number sense, but suggest a new look through the lenses of Bourdieus’ theories about habitus. Her agenda is different from the articles in TWG2. She claims that the definition of number sense in both meanings – innate and didactical – excludes particular groups of children, because it is only looking at the individual child’s success and therefore she reasons that a structural inequality is reproduces through the educational system.
The critic Turvill brings in is typical for many researchers in mathematics education. The question ‘How can we use results from neuroscience?’ seems unsolved. The discussion that Turvill brings up about the individual and the community is an important issue, but using results from neuroscience research in mathematics teaching does not have to cause only focusing on the individual child. What the neuroscience researcher offer is an ontological understanding of how the brain works when the topic is mathematics. How it could be taught is still left to the teachers.

Fostering number sense

In line with the didactical understanding of number sense some papers make suggestions how the development of number sense might be fostered. Ferreira and Serrazina (2011) summarize some of the main lines of argumentation in this context. Referring to Yang (2003, 132) they argue that “students‘ number sense can be effectively developed through establishing a classroom environment that encourages communication, exploration, discussion, thinking and reasoning” (p. 308).
The majority of papers tackling number sense closely associate number sense and mental calculation. However, the role of number sense related to mental calculation does not seem to be absolutely clear. On the one hand, it is argued that number sense is exhibited by

manipulating numbers mentally and by applying efficient mental calculation strategies. In this line of argumentation, flexible and adaptive mental calculation is regarded as a competence and number sense is regarded as one of the main influential factors on it (e.g. Rezat 2011). On the other hand, mental calculation is regarded as a way to develop number sense. In fact, studies sharing this line of argumentation investigate factors in the teaching and learning of mental calculation that have an effect on the development of number sense. E. g., Ferreira and Serrazina (2011) refer to studies that contend that the development of number sense is fostered if students are encouraged to formulate their own mental computation strategies.
A different approach is taken by Rechtsteiner-Merz & Rathgeb-Schnierer (2015). They use the notion "Zahlenblick", which has similarities with the didactical notion of number sense. In order to develop "Zahlenblick" they do not only focus on mental calculation, but generally suggest to "provide activities, which highlight problem characteristics, patterns and numerical relationships" (p. 355).

The scope of number sense in terms of number domains

Whereas the notion of number sense typically applies to the domain of natural numbers some papers within TWG 2 "arithmetic and number systems" also refer to number sense related to other number domains. This is also connected to research investigating students' mental computation strategies in these number domains. These authors' line of argumentation is that mental computation in other number domains will foster students' number sense. Therefore, the authors argue for promoting mental computation also in other number domains. The underlying general assumption is that number sense can be developed by mental calculation, i.e. is not innate. Therefore, the line of argumentation already indicates that the authors refer to a didactical notion of number sense.
The findings of Carvalho & da Ponte (2013) reveal that students develop particular mental calculation strategies in the domain of fractions. In this context, it has to be acknowledged that working in new number domains also poses semiotic challenges on students, because they have to understand a new number representation. E.g., if we look at the fraction $\frac{13}{17}$ it seems to consists of two natural numbers, but they are put on top of each other with a line in between. Furthermore, the value of the fraction, which consists of two natural numbers much bigger than 1, is actually less than 1. This is a semiotic challenge, which might interfere with conceptual challenges. Therefore, it is a challenge to distinguish strategies that indicate a different number sense in the domain of fractions from strategies that arise due to the particular challenges of the new number domain.
In contrast Rezat (2011) concludes in his investigation of students' mental calculation strategies with negative numbers that "all the problems were solved using familiar number-transformation strategies from the set of natural numbers. No strategies specifically related to rational numbers were observed" (p. 403). These opposite findings lead to the question if there is a general notion of number sense, which relates to all number domains or if there is particular number sense for every number domain. While Carvalho's findings support a hypothesis that a particular number sense related to fractions is helpful for mental calculation in this number domain Rezat's (2011) findings might be interpreted in the way that mental calculation in other number domains than the natural numbers draws on the number sense (of natural numbers) wherever possible.

According to Carvalho & da Ponte (2013, p. 284) "concerning fractions, number sense is related with the ability to use relational thinking […] to compare and work with this rational number representation and to understand the quantities involved". With her definition she adds relational thinking as an important feature of number sense in fractions and stresses the number representation and an understanding of rational quantities. All three features do not occur in the definition of McIntosh et al. (1992).

Drawing on the definition by McIntosh et al. (1992) Almeida and Bruno (2015) develop a framework, which characterizes number sense related to any number domain based on 8 components: "(1) understand the meaning of numbers; (2) recognize the relative and absolute size of numbers and magnitudes using estimates or numerical properties to make comparisons; (3) use benchmarks to estimate a number or magnitude when comparing or doing calculations; (4) use graphical, manipulative or pictorial representations of numbers and operations; (5) understand operations and their properties; (6) understand the relationship between the problem's context and the operation required; (7) realise that there are multiple strategies; (8) recognise the reasonableness of the problem (data, strategies and results)" (p. 238f).

The seminal definition of number sense provided by McIntosh, Reys and Reys (1992) does not make a number domain explicit while the innate number sense only applies to natural numbers. Consequently, the meaning of number sense in other number domains is not clear.

Discussion

The synopsis of research referring to number sense in CERME shows that number sense is a multi-faceted concept. Therefore, it is important to be clear about the actual notion of number sense that is referred to whenever the term is used.

Generally, there are two different notions of number sense: a notion of number sense from the biological/cognitive area and a different one from educational/pedagogical perspective. The majority of CERME papers refer to the latter perspective. The understandings and skills described in the didactical view of number sense are developed through teaching. In order to differentiate these two notions of number sense terminologically we suggest referring to the didactical view using the term 'number understanding' as opposed to 'number sense', which relates to innate abilities.

For us, the main question that arises from the multi-faceted concept of number sense is how to bridge the gap between the knowledge of number sense gained from the biological/cognitive area and the didactical view of number understanding. Why do we believe that this is important? Research in cognitive psychology and mathematics education developed important insights into the development of a conceptual understanding of numbers. While one area focuses on innate prerequisites of the development of such a conceptual understanding the other area is concerned with mathematical activities that foster this development. We believe that both fields can profit from each other. If on the one hand mathematical activities do not build on innate predispositions they might be idle. On the other hand, the aim of developing activities and learning arrangements in order to foster conceptual understanding of numbers might provide a direction for cognitive research.

Bridging the gap demands that the two different research communities, who use the concept 'number sense' differently, can communicate with each other and understand how

they can supply each other and use each other's research results, instead of being extremely critical or worse ignoring each other . Bridging the gap could be worthwhile because knowledge from educational neuroscience can complement the education with new knowledge that might change our beliefs and habits.
Dehaene's definition of number sense offers an ontological understanding of numbers as an innate skill. His research has inspired teachers to prepare and run several teaching episodes. For instance it is possible to work with big numbers already in kindergarten with children of 5-6 years. By asking the children "who has more?" it is possible for them to add and subtract big numbers without counting (Gilmore, McCarthy, and Spelke, 2007; Ejersbo, 2016). A longitudinal study shows that using the innate number sense regularly supports students' mathematical understanding (Halberta, Mazzocco, and Feigenson, 2008)
Further questions that arise from the synopsis of research on number sense in CERME are:

- Is number understanding a prerequisite for developing flexible and adaptive mental computation strategies or does the ability to mentally compute flexible and adaptively foster the development of number understanding?
- Is there an adequate notion of number sense/number understanding in other number domains than the natural numbers? How is the relation of number sense/number understanding and symbol sense or structure sense, respectively?

The gap between the biological/cognitive research in mathematics and the didactical research in mathematics can be overcome if we see it as an addition instead of comparison and it goes both ways. The students will be the winner in that case.

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