6 Argumentation and proof

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6.1 Introduction

The theme of this chapter has been the theme of one of the working group at CERME since the very beginning, being one of the main research themes of the community of Mathematics educators. However, since a long time Proof and proving has been a theme of debate not only in the community of math educators, but also in the community of mathematicians and in the communities of researchers in history and philosophy of mathematics (Thurston, 1994 ; Hanna, 1989, Hanna et al., 2010). Proof represents a very special case in respect to other math topics, like Geometry or Algebra, and though it may be considered a math topic in itself, it is intimately and specifically related to any Mathematics' filed. In the past years, the debate among the researchers has been very passionate, sometimes reflecting great divergences. This makes this theme fascinating but also witnesses to its complexity, manly in respect to the objective of outlining didactical implications that can be useful in school practice.

For this very reason, the first part this chapter will start with addressing some epistemological issues that since the beginning have been at the core of the participants' discussions and led to further elaboration of specific epistemological stances concerning the possible tension between argumentation and proof; the following part is devoted to one specific direction of research that emerged in the group and addresses the role of logic in argumentation and proof; finally, the third part will deal with the specific issue of teaching proof.

6.2 Historical, epistemological and theoretical issues

The variety of approaches and the diversity of the positions clearly appeared to be related to deep and almost implicit epistemological stands. Thus, not only epistemological, as well historical, issues have been addressed in the group discussions, but also happened that they were explicitly addressed and elaborated in the papers presented at the Conferences, as well in new version of them lately published, sometimes in collaboration with other members of the group. This section offers a critical synthesis of the contributions and tries to report on how specific issues emerged and evolved within the group.

6.2.1 The issue of terminology in search of a common ground

Since the very beginning of the Group life, it became clear that there was a need to share common meanings for the terminology we were using in our contributions. This same issue, as it was pointed in Mariotti (2006), was later addressed explicitly by Nicolas Balacheff (2008), who questioned the existence of a shared meaning of the term 'proof' and consequently of any other term or expression related to it.

Currently the situation of our field of research is quite confusing, with profound differences in the ways to understand what is a mathematical proof within a teaching-learning problématique but differences which remain unstated. (Balacheff, 2008, p. 501)

The first explicit approach to discussing terminological issues, and consequently epistemological issues, started at CERME 4 when the discussion on "The meaning of proof in mathematics education" was opened by David A. Reid who, in his paper, raised a key question "Is there any prototype of proof?". In line with the analysis carried out in Reid's paper, when confronted with specific examples, participants generally rejected certain arguments as mathematical proofs and accepted others, even though they were generally not able to define the characteristics of a prototype proof. Thus it seemed that regardless of the different epistemological positions that participants might have held, when asked to judge arguments as proofs or non-proofs experts find a common ground, but such a consensus seemed not to correspond to any specific shared set of explicit characteristics. As a consequence, it was less clear whether there was a common ground concerning teaching approaches towards proof; thus, the key question arose, whether researchers' epistemologies have any significant influence on the teaching approaches they espouse. An affirmative answer to this question was perhaps one of the first shared achievements of the group's discussions: this shared point led to a fundamental principle for a constructive discussion, consisting in asking the researchers for making their epistemology explicit. This led the group to a more fruitful discussion and, as we will show in the following, to progress in elaborating new tools of analysis.

6.2.2 Tension between epistemological and didactical issues

The introduction of an educational perspective induces a tension between epistemological and didactical issues that must find a resolution. An interesting example of a coherent position in this respect is the "genetic approach to proof" presented by Hans Niels Jahnke (CERME4). Starting from a specific and explicitly stated epistemological position, "to treat geometry in an introductory period as an empirical theory", Jahnke expounds a teaching approach that evolves from this epistemological position. The relationship between empirical evidence and proof gains a new meaning in the “genetic teaching approach” that is consistent with the background epistemological position explicated by Jahnke. Jahnke’s notion that there is a kind of dialectic or mutual support between intellectual proofs (to use a term of Balacheff’s (1999) to identify the case when arguments are detached from action and experience) and recourse to data (pragmatic proofs) throws a new and refreshing light on their relationship and on the seemingly negative findings of Fischbein and Kedem (1982) and Vinner (1983). In section 3 a specific outline of the evolution of the issues arising within the Group from a teaching and learning perspective will be developed.

According to different research foci, it became clear that making explicit epistemological positions was not sufficient, it was necessary to elaborate them according with specific suitable tools.

6.2.3 The elaboration of specific epistemological stances

The shared concern about making explicit one’s own perspective and assumption related to the use of a certain terminology was firstly aimed at improving mutual understanding, however it was slowly sided by the elaboration of specific epistemological stances into theoretical tools for framing research studies on teaching and learning proof. Such tools were borrowed from different research fields. First examples can be found in some of the papers presented at CERME 5, and developed in the following.

Pedemonte , Mariotti & Antonini (CERME 5) exploit specific notions , such as Cognitive Unity (Boero et al., 2000; Pedemonte, 2002) and Theorem (Mariotti et al. 1997), emerging from the elaboration of Toulmin’s model for argumentation to describe and compare argumentation processes occurring in producing conjectures.

Stylianides & Stylianides (CERME 5) bring to the attention of mathematics education researchers a rich body of psychological research on deductive reasoning, related to the well-known paradigm of mental models (Johnson-Laird, P.N., 1983).

Other contributions concern the specific role of tools of analysis coming from Logic; this specific case is dealt in the section (6.2) of this chapter.

We intend to focus on two specific cases that we consider exemplar of the development of the group discussion: the case of Toulmin’s model for argumentation and that of Habermas’ notion of rationality, they were developed starting from the construct of Cognitive Unity.

The case of the notion of Cognitive Unity

The epistemological and cognitive analysis carried out by Duval (1991; 1992-93) had warned us that the main issue from an educational perspective was exactly in the proximity of the two processes and in the danger of mistaking one to the other, thus the need of a careful distinction between them, stressing the theoretical nature of proof and even disvaluing argumentation as a possible obstacle to the developing of a sense of proof.

Of course this radical position taken by Duval leaves place for a wide range of intermediate possibilities and variations in conceiving the relationship between argumentation an proof, however it focuses on specific features of a mathematical proof that characterize it and for this very reason can be hardly neglected without a serious/great loss for mathematics education. (Mariotti, 2006)

The notion of Cognitive Unity, firstly introduced in the Topic Working Group discussion by Bettina Pedemonte at CERME 3, offers one possible direction of investigation to resolve the tension between epistemological and didactical issues. Elaborating on a previous definition (Boero et al. 1996), the main objective of this theoretical construct was that of providing a tool of analysis to investigate the possible gap between argumentation and mathematical proof, focussing on the argumentation process supporting a conjecture and the consequent proof produced to validate the final statement. Cognitive unity is based on a twofold analysis carried out considering conjectures in terms of "their contents (some words, some expressions, the theoretical framework if it exists in the argumentation, and so on), and/or their structure (abductive, inductive, deductive and that takes on the one hand the point of view of the system of reference, and on the other hand the structure (abductive, inductive, deductive and so on)." (Pedemonte , CERME 3). The first and the following contributions presented by Pedemonte at CERMEs, and later on published in (Pedemonte, 2007), were centred on the structural analysis, where the main theoretical tool used was borrowed from Toulmin and adapted to the specific case of producing conjectures as answers to open problems. The effectiveness of Toulmin's model in analysing the complex structure of an argumentative chain is not limited to highlight the possible gap between argumentation and mathematical proof it also provide a powerful tool to discriminate between different types of arguments. This is witnessed by the appearance of an explicit reference to using the Toulmin's model in papers presented in the successive CERMEs, so that at CERME 7 we see a number of contributions integrating this model in their theoretical framework. For instance,

Jenny Christine Cramer develops a methodology combining Toumin's scheme and a collection of topical schemes with an epistemic action model in order to shed light on the relations between argumentation and knowledge construction. (J.C. Cramer, CERME 7).

While the Toulmin's model was showing its effectiveness, its limits emerged too. It became clear how in some cases, students met difficulties inherent in the lack of "structural continuity" when they have to move from creative ways of finding good reasons for the validity of a statement, to their organization in a deductive chain so that to be an acceptable proof. Thus, beyond becoming aware of the difficulties, there was the need of describing the difficulties in the proving process interpreting their origins, in order to investigate how these difficulties could be overcome.

At CERME 6, Morselli and Boero (CERME 6) presented the analysis of some examples through the model of Habermas in the special case of conjecturing and proving and show the viability and usefulness of such a model for that purpose. Habermas' model, conceiving rational behaviour on the based of the relationships between teleological, epistemic and communicative rationality, seemed to offer a fruitful tool of analysis for the dynamics of students' arguments. The suggestion of integrating Habermas' model was followed by some of the researchers that attending CERME 7. For instance Patricia Perry, Oscar Molina, Leonor Camargo, and Carmen Samper presented a paper where they analyse the proving activity of a group of three university students solving a geometrical problem in a dynamic geometric system. Solving the problem requires formulation of a conjecture and justification in a theoretical system. For their analyses, the authors refer to the integration of Toulmin's and Habermas' models to elaborate / define components of a successful performance.

Cognitive and meta-cognitive issues were specifically developed in three papers presented at CERME 7 in line and deepening the research in this perspective. The paper by Paolo Boero and the one by Ferdinando Arzarello and Cristina Sabena draw explicitly on the integration of the models of Toulmin and Habermas that was presented at CERME 6. In particular, Arzarello and Sabena go further and used the idea of "meta-cognitive unity" in order to give reason of success and difficulties in indirect proofs.

The paper presented by Boero at CERME 9, is illuminating in order to understand the genesis of the integration between the two theoretical tools: the notion of cognitive unity analysed through the Toulmin's model and the Habermas' approach based on the three components of rationality. The paper starts with the story of a protocol and the problems that the complexity of its interpretation posed to the researcher: how to interpret the apparently chaotic argumentation process supporting the validation of a quite obvious statement. A mere description in terms of its structure according the Toulmin's model was not enough to do justice to the complexity of the student's mental process. As the author writes "The [first] need suggested us to try and adapt Habermas' construct of rational behaviour to Ivan's problem solving - as a process driven by intentionality to get a correct result by enchaining correct steps of reasoning, and to communicate it in an understandable way in a given community" (Boero CERME 9, pag. 96).

The analysis of the proving process through the notion of cognitive unity and its elaboration through the Habermas' model of rationality has interesting implication on the teaching and learning perspective. Indeed, Morselli and Boero (CERME 6) claim the possibility of dealing with the approach to theorems and proving in school as a process of scientific "enculturation" (Hatano & Wertsch, 2001) consisting in the development of a special kind of rational behaviour, according to that described by the Habermas' model.

6.3 The role of logic in argumentation and proof

The role of logic in Argumentation and Proof is a rather controversial issue in the wide world research community in mathematic education (Durand-Guerrier et al. 2012). From CERME 3, some papers refer explicitly to logic according to three different perspectives, closely intertwined with epistemological issues. In CERME 7, although no paper was explicitly devoted to logical issues, the logical aspects of proof, the way of taking into account everyday logic competencies in class and of considering the role of semantics aspects and the place for logical matters in the teaching and learning of proof and proving had been discussed. In continuity, four papers referring explicitly to logical issues were presented in CERME 8. We present briefly the two main aspects that were discussed along the CERME conferences.

6.3.1 Search for a logical reference for analysing proof and proving

The search for a logical reference for analysing proof and proving in mathematics from an educational perspective was motivated by the fact that as some psychological studies seemed to show that formal logic is not a model for how people make inferences, the idea that logic was useless for developing reasoning skills (e.g. Johnson Laird 1986) was rather popular among mathematics educators and researchers. Durand-Guerrier (CERME 3 & 4) uses the model-theoretic approach introduced by Tarski and distinguishes three dimensions: syntax (the linguistic form), semantic (the reference objects), pragmatic (the context, and the subject's knowledge in the situation), for a didactic analysis of mathematical reasoning and proof. She considers these distinctions as important in order to foster argumentation and proving processes of students and claims that "natural deduction" is a powerful tool to analyse proofs in a didactic perspective, for teachers as well as researchers. She argues that the model-theoretic approach introduced by Tarski calls for continuity between argumentation and proof, in contrast with the discontinuity seen by researchers working in a cognitive approach. This point has been discussed anew in the paper by Barrier et al. in CERME 6, opening a discussion with papers whose main reference was Toulmin, such as Pedemonte's one (Pedemonte 2007). The authors supported the importance of taking into account the distinction between truth and validity, and this issue emerged again in CERME 8 with the paper of Mesnil.

A second perspective concerns logical concepts such as implication, which plays a central role in proof and proving, and is known to face students with strong difficulties. Deloustal Jorand (2007) analyses mathematical implication from three different points of view: formal logic, deductive reasoning, and sets. A didactical engineering, based on the assumption of the necessity of making these points of view interact is described carefully and its implementation discussed, showing how a suitable situation can raise the issue of implication.

6.3.2 Relationships between logic and language in proof and proving

The relationships between logic and language in proof and proving have been also a recurrent theme that has been widely discussed in CERME 8 with 5 papers focusing on this theme (Cramer 2013, Mesnil 2013, Azrou 2013, Chellougui and Kouki 2013, Muller-Hill 2013)

The main points that emerge from the discussion are: identification, in the relationship between logic and language, of aspects that are likely to be an obstacle for developing proof and proving skills, and of aspects that are likely to favour it; the interest of teaching logic for fostering proof and proving competencies; the interest for teachers, of logical analysis in mathematical discourse, and how to do it; the relationships between logic and formalisation. The following questions were discussed during the sessions without reaching a consensus: Should we consider logical competencies and/or logic as a body of knowledge; logic as a theory modelling human reasoning and/or as a theory aiming to control validity of proof? Should logical proof be considered both in terms of a final product and as a process in action? Is it relevant to teach logic at secondary school or not?

Two main lines emerged from discussions. The first one concerns the role of logic and language as a possible tool for researchers and the implication that research findings could have for teachers. At first logical analysis of a statement appears as a fecund means to deepen and enrich an *a priori* analysis of a task, e.g. by enlightening possible unexpected ambiguities that could impend the understanding of the mathematical statement at stake or by favouring the identification of didactical variables in order to enlighten possible choices for the study of a given concept. In addition, logical analysis of mathematical discourse can be used to analyse students' production in order to better understand them and in some cases open to new interpretations. Last but not least, logical systems such as natural deduction provide the researchers a powerful tool to check proof validity. The second one concerns the relevance, or not, of teaching logic in order to foster proof competencies, and in case of a positive answer how to do this.

This emergence of logical issues in European research on argumentation and proof discussed in the ERME conference encountered interest of non-European researchers in the frame of the 21st ICMI study on Proof and Proving in Mathematics Education, where several papers on this topic were submitted. As a consequence, a chapter was devoted explicitly to this question in the conference book (Durand-Guerrier et al. 2012). Three active members of the CERME working group on argumentation and proof were involved; a US and a Canadian colleague completed the authors' team. In this chapter, the relevance of and interest in including some instruction in logic in order to foster competence with proof in the mathematics classroom is examined. Considering the contradiction between two assumptions: doing mathematics at the secondary level in itself suffices to develop logical abilities, on the one hand; many tertiary students lack the logical competence to learn advanced mathematics, especially proof and other mathematical activities that require deductive reasoning on the other hand, the authors support the claim that it is necessary and possible to introduce in the curriculum activities aiming at explicitly developing the logical competences required by advanced mathematical activity.

6.4 Teaching of Proof

This section will focus on issues of proof in the classroom, both from the point of view of the teacher and from the point of view of the students. We use the term "classroom" broadly to denote formal learning settings at both the school mathematics and the university levels, including teacher education.

Issues of teaching proof started to receive more attention in the discussions of the Thematic Working Group (TWG) on "Argumentation and Proof" during the last decade of the ERME conferences. For example, in their introduction to the TWG papers of CERME 6, the working group leaders noted: "No great discussion on didactic issues related to proof can be found in the contributions to the working group. The only exception is the specific example of a teaching intervention presented in the paper of Douek." (Mariotti et al., 2009, p. 179). After the discussions that took place during the TWG sessions in CERME 7, the group concluded that more research and discussions during upcoming ERME conferences were needed on the teaching of proof, especially in relation to designing activities that could be used to foster "argumentation and proof skills along the curriculum from kindergarten to university" (Durand-Guerrier et al., 2011, p. 96).

The epistemological and terminological issues we discussed previously in this chapter received more attention in the earlier discussions of the TWG. This is not surprising as it was sensible for the group to try to clarify first epistemological and terminological issues before focusing on the design of classroom interventions to promote school and university students' understandings of proof. We do not consider, of course, that epistemological or terminological issues have now been clarified completely, in the context of the ERME conferences or in the field of mathematics education more broadly. Yet, the evolution of the issue of the design of classroom interventions in the area of proof within the TWG mirrors its general evolution outside the group in the mathematics education community and was influenced by developments in the field in relation to (1) terminological and epistemological issues and (2) theoretical constructs and frameworks that helped explain classroom phenomena. We exemplify each of these points in the next two paragraphs.

In relation to terminological issues, the issue about the meaning of proof that was raised by David Reid in CERME 4, and before that by Balacheff (2002), was taken up by Andreas Stylianides who presented a conceptualization of proof in CERME 5 that was published later that year (Stylianides, 2007). This conceptualization was used in various classroom-based research studies in the area of proof. For example, the conceptualization constituted the basis for a classroom intervention in the area of proof at the university level that we discuss below (Stylianides & Stylianides, 2009). For another example, Reid and Vargas (2017) in CERME 10 are presenting a proof-based teaching intervention, organized as a 3-year design experiment, that is consistent with the meaning of proof in Stylianides (2007) and aims to help third-grade students develop their knowledge of division of natural numbers.

In relation to theoretical constructs, in CERME 4 Jahnke (2005) used the metaphor of a *theoretical physicist* to explain common student thinking about the meaning of empirical verification in the proving process, arguing that this student thinking occurs with a certain necessity and is a consequence of a meaningful behavior. Jahnke then continued to propose a didactical approach, which he further elaborated in Jahnke (2007) and illustrated in the domain of Geometry. In this approach, Geometry is treated in the early stages as an empirical theory that bears similarity with the natural sciences, such as Physics. The didactical approach is "centred around the idea that inventing hypotheses and testing their consequences is more productive for the understanding of the epistemological nature of proof than forming elaborate chains of deductions. […] In this approach proving and forming models get in close contact." (Jahnke, 2007, p. 79) This didactical approach together with other similar approaches, in which experiments at the school level are combined with theoretical aims, discussed at ERME conferences (e.g., Bartolini-Bussi, 2010) and beyond (e.g., Boero et al., 2007), informed the classroom-based intervention study by Jahnke and Wambach (2013) which we discuss below.

In the rest of this section, we will discuss three studies that aimed to promote students' understandings of proof. The first will be Douek's (2009) study as an example of an early ERME paper that focused on issues of teaching in the area of proof. The other two will be the two intervention studies we mentioned in the previous two paragraphs (Jahnke & Wambach, 2013; Stylianides & Stylianides, 2009) as examples of studies that built on ideas discussed in earlier ERME conferences. For a more detailed discussion of classroom-based interventions in the area of proof the reader can refer to Stylianides et al. (2017).

Douek (2009) presented a theoretical approach that could be used with lower secondary students to help them develop an awareness of some important features of proving theorems. The approach was then exemplified in a task sequence specific to the Pythagorean theorem. The main idea of the approach is to guide students' work through engaging them in conjecturing activities and guided proof construction, and finally helping them through discussion and "story making" to focus on important characteristics of the organization of the proof. The latter can help students make sense of the links between statements or arguments in the proof.

Jahnke and Wambach (2013) conducted an intervention involving eighth-grade students in Germany, which aimed to help develop students' understanding that proofs are based upon certain assumptions. The intervention took place during 8 Geometry lessons and was situated in the attempts of ancient Greeks to model the so-called "anomaly of the sun." The students were asked to assume that they had available to them the methods and tools that were known to ancient astronomers at the time. These restrictions were similar to the restrictions imposed on the Cabri tools available to the students in another classroom-based intervention discussed by Mariotti (2012, 2013) and were an important factor that contributed to students becoming more conscious of the role of assumptions in building a deductive theory.

Stylianides and Stylianides (2009) reported on a classroom-based intervention that they developed in a 4-year design experiment in an undergraduate mathematics course for prospective teachers in the United States. The intervention lasted less than 3 hours and aimed to help students begin to overcome a widespread misconception in the area of proof: that empirical arguments offer secure methods of validating mathematical generalizations. In more detail, the intervention aimed to help students see an "intellectual need" (Harel, 1998) to learn about proofs and involved the implementation of a purposefully designed task sequence that motivated and supported stepwise progressions in students' knowledge about proof along a pre-specified learning trajectory. This trajectory began with a naïve empirical conception (Balacheff, 1988), moved on to a crucial experiment conception (ibid), and ended to the desired non-empirical conception.

6.5 Final remarks and conclusions

As final remarks, we would like to mention some of what we consider good practices that in the years have been implemented in the organization of the TWG and that we think have been effective for the development of the Working Group as a research community.

Participation to the group has been rather stable, that means that a number of participants attended the TWG activities for many years, so that we can really think of the group as a community where researchers that know each other's work and not only support but also integrate the different theoretical perspectives that are presented and discussed.

For some of the youngest among us, participation started during or immediately after their doctoral experience, and continues till now, also serving as group co-leaders, it is the case of Samuele Antonini and Bettina Pedemonete, but also Kirsti Hemmy and Christine Knipping. Some of the contributions that were firstly presented at CERMEs, were subsequently published in international journals, so that we can reasonably think of the positive effect on these studies of the group discussion both on specific and on general issues. As a matter of fact, though not explicitly mentioned as sprouting form CERMEs' experience the ZDM special issue, edited by Balacheff and Mariotti (2008) collected papers on the theme of "Argumentation and Proof", and most of the authors who have been participants of the TWG, contributed with elaborations of their CERME's papers.

Besides the general policy of the WG, recommending the participants to read the papers in advance in order to minimize the time for the presentations and maximize the time for the discussion, the leader and the co-leaders carried out an organization work - sometimes more complex of what could be foreseen – in order to group the presentations and consequently the discussions according to specific research perspectives; sometimes, the efficacy of the discussion was fostered by asking the participants to act as discussant, preparing specific questions to stimulate the debate. This has been particularly effective to create a collaborative attitude. Similarly, when time allowed, after the discussion of the papers, the working group participants split into small groups, with the objective of deepen specific theoretical or methodological aspects that emerged from the general discussion. Traces of these discussions can be found in the introductions to the Working Group contributions, published on the Proceedings. For instance, at CERME 5 the following set of issues raised by the debate were discussed:

- Using formal models for investigating proof and comparison between different models for investigating proof.
- Proof in the classroom: focus on the tasks, focus on the mathematical domains, focus on the teacher.
- Teaching experiments for investigating proof: methodological issues related to investigating proof in the school context.

Of course the size of the Working Group is determinant for a good functioning of the debate, but in spite of the possible organization difficulties this practice has been highly rewarding, and definitely contributed to build up a community of researchers.

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