History and Mathematics Education

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Abstract

When considering mathematics education from a historical perspective one may follow two distinct strands: history (as a subject or a didactical tool) *in* mathematics education; and history *of* mathematics education. Within ERME both strands are represented since CERME-6 (Lyon 2009). We sketch goals, methods and structure of both research strands as well as their main results while providing illustrative examples of both strands from the work within the CERME thematic working group since CERME-6. At the same time we take into account the cooperation between researchers with different backgrounds, between experienced researchers and young researchers as well as by relating to the general communication between the researchers in the group.

Introduction

CERME's thematic working group on "History in Mathematics Education" is a rather new initiative - historically speaking. We will first review "the pre-ERME period", as far as history in mathematics education is concerned, and then describe the creation of the CERME group. Its work from CERME-6 to CERME-9 will be represented in two ways. First we outline the contents of four representative papers, which will show the the various theoretical frameworks and methodologies applied in the the group. Also the collaboration between young and established researchers can be recognized in this subset of papers. Then the distribution of the papers along the research questions set by the organizers is given in a table.

One specific feature of the group has been the group thematic discussions, which, starting from the contributions of the participants, addressed various general issues related to conducting research in the two strands. While the history *IN* mathematics education strand addresses actual uses of history of mathematics in actual teaching and learning, the other strand addresses the very history *OF* mathematics teaching and learning. We will argue that this distinction and the thematic discussions of both strands ended up being a strength of the work carried out in the group, not least in terms of communication and collaboration.

The pre-ERME period

Early interest in the connection between the history of mathematics and mathematics education goes back to the nineteenth century. Fauvel (1991) and Tzanakis and Arcavi (2000, p. 202)

present some early cases. Fauvel quotes George Heppel, who in 1893 presented three conditions for “The use of history in teaching mathematics”: (1) history should be auxiliary, (2) history should assist the student in learning mathematics and (3) history should not be examined. Heppel feared that history would overload the ‘ordinary schoolboy or schoolgirl’, a fear that was also expressed by many later authors. A broad perspective was sketched by Otto Toeplitz, who introduced the so-called genetic principle. Toeplitz (1927) pleaded for taking the historical development of mathematical ideas as a guide for teaching. Jahnke and Fried (2015) describe and translate (Toeplitz 1927).

History of mathematics, either as a topic to be taught or as an inspiration and background for mathematics teaching, became *en vogue* in several countries in the 1960s and 1970s. Widely distributed and influential was the 31st NCTM yearbook on “historical topics for the mathematics classroom” (1969, updated 1989). The Preface describes it as “a pioneering effort to assist in the teaching of mathematics from a historical perspective”.

The next phase can be characterized as a period of combined efforts. Joint work of teachers and academics within the French IREMs (*Instituts de Recherche sur l’Enseignement des Mathématiques)* was brought to the UK thanks to Evelyne Barbin and John Fauvel. IREMs were set up in the 1970s for research into mathematics teaching in France, which was carried out by practising teachers (more in Fauvel 1990, p. 139). After 2000 the question how the actual teaching with historical material went became more and more important, also due to the influence of ERME. The introduction in the UK started with a conference of the *British Society for the History of Mathematics* (BSHM) in 1998 in Leicester. Fauvel organised in 1990 the first HiMEd (History in Mathematics Education) conference, for which he made the French IREM materials available in English (Fauvel, 1990). This conference led to a special issue of *For the Learning of Mathematics* (vol. 11/2, 1991) and several other publications. Until 1996 there were HiMEd-conferences of the BSHM every other year (Nottingham 1992, Winchester 1994, Lancaster 1996).

The HPM community

International collaboration had started at the 1972 second *International Congress on Mathematical Education*. ICME-2 staged a working group on ‘History and pedagogy of mathematics’. In 1976 ICME-3 had three sessions about ‘History of mathematics as a critical tool for curriculum design’. The participants proposed to have regular sessions at future ICMEs on history and pedagogy of mathematics. The executive committee of ICMI decided to start with the *International Study Group on Relations between History and Pedagogy of Mathematics*, or just the HPM Study Group. HPM returned at all ICME congresses since 1980. From 1984 there also were special HPM satellite conferences before or after the main ICME. HPM publishes a Newsletter since February 1980, which is sent out via a network of national distributors. In 2016 it is approaching issue 100. As HPM has no central administration, the newsletter is its binding structure. Fasanelli and Fauvel (2006) give a fuller history. A survey of conferences is given on http://www.clab.edc.uoc.gr/HPM/about HPM.htm .

History *in* mathematics education as a research area

Publications about research studies on history in mathematics education gradually became more present. Journals brought thematic issues, such as *For the Learning of Mathematics* (Fauvel, 1991), *Educational Studies in Mathematics* (Radford et al., 2007), and *Science & Education* (Katz et al., 2014). Collections of papers published in the MAA-notes illustrate well the scope of the field (e.g. Katz & Tzanakis, 2011). An overarching view of the field by the year 2000 is given in the ICMI-Study (Fauvel & Van Maanen, 2000). A comprehensive account of empirical studies on history in mathematics education is available in (Jankvist, 2012), which also provides an overview of the academic fora of the field, such as the *European Summer Universities on the History and Epistemology of Mathematics Education* (ESUs) as well as the CERME working group.

History *of* mathematics education

Interest in the history of mathematics education goes back to the 19th century. In the *Handbook on the History of Mathematics education* (Karp & Schubring 2014) Schubring reviews the historiography of teaching and learning mathematics. He locates early interest especially in Germany, with books about one particular Prussian Gymnasium (1843) and about methods for teaching arithmetic (1888). By the end of the century historical studies also appeared in several other countries.

An important stimulus came from international cooperation within the *Internationale Mathematische Unterrichts-Kommission* (IMUK, 1908), since 1952 continued as the *International Commission on Mathematical Instruction* (ICMI). IMUK and ICMI ordered several broad international studies, which generally concerned parts and aspects of the mathematics education of that period. Schubring also presents a number studies about mathematics education in specific countries (Germany, England, USA and Canada, Finland, Russia). The *Handbook* can be considered as the culmination of these earlier activities and as a result of cooperation within the present day research community.

History *of* mathematics education as a research area

Research about the history of mathematics education was first put on the agenda of ICME-10 (Copenhagen 2004). The goals then were to “gather the researchers working in this field [...] and develop research programmes, which enhance international perspectives and the study of the ‘general’ within national specific histories.” The initiative connected to existing national activities, and had a considerable follow-up. A specialized *International Journal for the History of Mathematics Education appeared* as of 2006, but publication terminated in 2016. Biennial conferences started in 2009 (Reykjavik, base of the important initiator Kristín Bjarnadóttir), followed by Lisbon (2011), Uppsala (2013), Turin (2015) and Utrecht (planned for 2017). Also, the participation at ERME-conferences since CERME-6 much stimulates researchers in the field.

The creation of the ERME thematic working group

The creation of the CERME group should be seen as an attempt to create a forum and a platform for fostering empirical studies in the field of history in/of mathematics education. This is also reflected in the bullets of the "call for papers/posters" that the group has operated with since CERME-6:

1. Theoretical, conceptual and/or methodological frameworks for including history in mathematics education;
2. Relationships between (frameworks for and empirical studies on) history in mathematics education and theories and frameworks in other parts of mathematics education [[this point featured only from CERME-7 onwards]];
3. The role of history of mathematics at primary, secondary, and tertiary level, both from the cognitive and affective points of view;
4. The role of history of mathematics in pre- and in-service teacher education, from cognitive, pedagogical, and/or affective points of view;
5. Possible parallelism between the historical development and the cognitive development of mathematical ideas;
6. Ways of integrating original sources in classrooms, and their educational effects, preferably with conclusions based on classroom experiments;
7. Surveys on the existing uses of history in curricula, textbooks, and/or classrooms in primary, secondary, and tertiary levels;
8. Design and/or assessment of teaching/learning materials on the history of mathematics;
9. The possible role of history of mathematics/mathematical practices in relation to more general problems and issues in mathematics education and mathematics education research.

For a discussion of the coming about of the group, please refer to (Jankvist et al., 2011) and for a list of the number of papers, see table 1 below. As can be seen from bullet 9 above, due to CERME's focus on present day mathematics education, a constraint was made that studies on the history of mathematics education should relate their historical results to current teaching and learning practice. Roughly speaking, the CERME group counts around twenty participants, but not always the same twenty people. Also, it is a rather mixed crowd. There is an overlap with the HPM community, but the group has also managed to attract newcomers. Some participants come from the history of mathematics while others are more rooted in mathematics education or mathematics proper. In hindsight, the group has been rather successful in combining young and established researchers. One of the main things expressed when the group has been evaluated is its friendly, inclusive and productive atmosphere, where everybody talks to and interacts with everybody. In 2011 one young participant said:

> "A week ago ... I didn't know how everyone in the WG would react to my work and my opinions (if I had enough courage to express them). Today I have in my memory

the best conference I ever attended: a fantastic working group that made me desire for more opportunities to work with everyone."

For the following two sections we selected four papers from the output of the group. The first two concern aspects of history *in* mathematics education, while the last two concern the history *of* mathematics education.

Selected papers on history *in* mathematics education

Paper 1: Development of learning strategies and historical awareness

Kjeldsen (2011), whose background is in the history of mathematics, suggests a theoretical framework for discussing "how history benefits students' learning of mathematics, and develops students' historical awareness" (p. 1700). She draws on a framework by the Danish historian Bernard Eric Jensen (2010) in which he outlines a broad approach to history. Her point is that when using history of mathematics in the teaching and learning of mathematics a didactical transposition is needed; the mathematics studied in school is not studied in the same manner as that at the university.

Jensen (2010) distinguishes between several approaches to history: a pragmatic vs. a scholarly approach to history; observer history vs. actor history; identity neutral vs. identity concrete history writing; and finally, a so-called "living history" approach. Kjeldsen addresses mainly the first two pairs. In a pragmatic approach the use of history is guided by the idea that through history we are to gain knowledge of the world today. History is studied from a utility perspective. This is contrasted to historians' critical distance to past events in a more scholarly approach. From another perspective, if, on the one hand, history is used to to orient oneself and act in a present context, then it is referred to actor history and history helps to intervene. If the past, on the other hand is used retrospectively with a purpose to enlighten rather than to act or intervene, then Jensen refers to it as a use of the past from an observer perspective.

As an illustration, Kjeldsen argues that if the focus is on developing students' mathematical competencies (e.g. Niss & Højgaard, 2011), a pragmatic approach from an actor perspective may be considered. However, mathematical competencies may also be developed alongside the development of historical overview and awareness, in which case the weight should be on a scholarly approach, e.g. with an observer perspective. The paper ends with an empirical example of an in-service teacher who "used different approaches to history and used past episodes from various perspectives for different purposes" so that history "was used in ways in which students gained genuine historical insights, developed learning strategies, and enhanced their mathematical problem solving skills even though they worked on mathematics that might not be part of the core curriculum" (p. 1708).

Paper 2: Teaching the concept of tangent line using original sources

Mota, Ralda and Estrada (2013) report about a teaching experiment in two consecutive years of secondary school (grades 11 and 12), in which the concept of tangent line is part of the mathematics curriculum. The authors wanted to determine the benefits and disadvantages of the introduction of the concept via the discussion of historical texts, chosen from Greek authors via 17th century French mathematicians and Leibniz to a 1790 Portuguese textbook. In Grade 11 21 students worked in groups on tasks about five texts from antiquity to Leibniz. In each text the tangent was defined. Typical tasks for students were: to answer the question “Is Euclid’s definition suitable for Archimedes’ spiral?” and “Using Fermat’s method, determine the extremes of a 2nd degree polynomial function.” In grade 12 half of the group of 22 students had also participated in Grade 11. They appeared to be more receptive for the digestion of the different historical definitions via notes that each students wrote as a preparation for a classroom debate.

The paper is a fair example of the transitional phase in which studies about history (as a tool) in mathematics education were in 2013. Teacher and researcher, who often coincide in one person, have a strong positive attitude towards using history in mathematics education. In Mota’s paper this is reflected for example in the paper's final paragraph, which starts with the words “it is our conviction …”. Also the teaching material receives a clear and complete display. But no real evidence is presented to support the conviction that contact with the historical evolution of the tangent was helpful for students. Instead of presenting evidence, the authors give a list of the obstacles that they encountered during the experiment (for example, students arguing that mathematics is not history). The next step in the development of a paper like this would be to pay more attention to the learning of the students, and to step from "conviction" to the “evaluation of evidence”. This paper is part of a larger Portuguese PhD project. It may well be that a later phase of the project will concentrate more on evidence about the learner. That would be much in line with the goals of ERME.

Selected papers on history *of* mathematics education

Paper 3: Arithmetic in Brazilian primary school at the end of nineteenth century

Da Costa’s (2009) paper shows a valuable feature of the CERME conferences: CERME assembles researchers from a variety of cultural and national backgrounds. The author was a doctoral student from Brazil, who worked for a year in Paris in a collaborative team of doctoral students. The Brazilian-French connection is clearly recognizable in the structure of the paper, which researches Brazilian arithmetic teaching around 1900 from a French perspective, especially Chervel’s 1998 framework of the history of school disciplines.

Central in Chervel’s framework is the *vulgata*, the standard set of educational principles and materials shared by most of the teachers. Part of the vulgata are representative teaching materials. For arithmetic education in Brazil da Costa describes the *Cartas de Parker* (Parker’s

Cards, named after Francis Wayland Parker, 1837-1902) and their introduction in Brazil. Parker connected with his cards, on which numbers and relations between them were represented geometrically, to the ideas of Grube and followers in Germany. The introduction of the cards in Brazil was much stimulated by a series of positive articles in a pedagogical journal. The teaching method was heuristic, discovery by the pupils was stimulated and the strict order of teaching (addition → subtraction → multiplication → division) was given up, as was the importance of memorizing.

The paper is a fair example of the power of crossing borders: national borders in the 19th century and also in the research community of the 21st century.

Paper 4: The first periodical on mathematical sciences in Ottoman Turkey

The paper by Alpaslan, Schubring and Günergun (2015) is an example of a collaboration between a young PhD-student (Alpaslan) and two experienced researchers. They discuss the periodical, *Mebahis-i İlmiye*, which was published from 1867 to 1869 in Ottoman Turkey. The purpose of the periodical was to provide mathematical education to society, when the Ottoman empire was involved in several wars. The periodical presented a wide variety of pure and applied mathematics, often related to physics, engineering, including also vocational mathematics.

The specific questions that the authors seek to answer are: (1) For what reasons did *Mebahis-i İlmiye* provide the 19th century Ottoman Turkey society with mathematics education? How were the reasons addressed in its content? (2) On which mathematical traditions did the periodical rely? These questions are addressed through Niss' (1996) three categories of fundamental reasons for mathematics education: (i) the technological and socio-economic development of society; (ii) the political, ideological and cultural maintenance and development of society, and finally (iii) providing individuals with prerequisites that may help them to cope with life in general. Another theoretical construct used by the authors is that of transmission of mathematical knowledge as a dissemination process of mathematical ideas from the scientifically established 'metropolis' countries to the not yet scientifically productive countries in the 'periphery' (Schubring, 2000):

> Findings indicate that *Mebahis-i İlmiye* addressed all the three kinds of reasons for mathematics education (Niss, 1996) to a certain degree. The authors utilized transformation of the recent knowledge of both pure and applied mathematics from Europe, mainly from France, as the 'metropolis' of the time (Schubring, 2000). Reception occurred in the difficult social setting of conflicts between modernizers and traditionalists, and within the already existing culture of Islamic mathematics. An important aspect of this transmission was the development of a terminology for the modern mathematics in Ottoman Turkish language, since the traditional mathematics did not provide terms for the new developments in the field. (Alpaslan, Schubring & Günergun, 2016, p. 1788)

Development of topics in papers and posters

We will now try to indicate what topics received the main interest from CERME-6 to CERME-9. The answer comes through categorizing and counting by one of the authors. The data are presented in Table 1. The rows correspond with the subdivision in the call for papers, discussed above. Point 9 covers the history OF mathematics education and is here subdivided in four categories, represented in rows 9 to 12. A paper may belong to several categories. In that case we distribute (not necessarily equal) fractional scores of the paper over the respective rows; if a paper mainly belongs to one category but also to two subsidiary categories, the scores for these three categories can be 0.5, 0.25 and 0.25. The number 1.05 in cell (CERME-9, row 5), for example, consists of scores for three papers: 0.5+0.3+0.25.

The numbers are small and they are based upon the judgment of one person only. Handle with care, seems the proper approach. A first observation is that one third of the papers and also one third of the posters are about the history *of* mathematics education, against two thirds for the history *in* mathematics education.

The table also reflects the methodology of the historian. The difference between the data in rows 9b and 9c, for example, (0.9 and 5.12 papers) can be explained from the fact that it is more difficult to study historical teaching practice than the teaching and learning materials. There is little information about how one taught, whereas the materials survived the decades or even centuries. The main interest went to historical studies about mathematics education as a social, cultural and also political phenomenon (6.57 papers in total).

In the domain of history *in* mathematics education the interest in teaching materials is also substantial (5.68 papers). Research about theoretical frameworks for including history in mathematics education had about the same number (5.53) of papers. The two categories that drew most attention are those that study the way history is applied in primary, secondary and tertiary education (6.85 papers) and in teacher education (6.12 papers).

A closer look to the papers or posters behind the numbers reveals that 4 of the total number of 11 posters came from a Portuguese group of PhD students, who in 2011 attended CERME-7, and who worked more or less as a team on comparable research topics. Such a joint contribution has a major influence on the relative importance of the data. The table does not reveal important aspects of research such as the chosen methodology, or a possible research paradigm or to what extent a paper focuses on the learner. For these aspects we have to refer the reader to the fuller accounts of the four selected papers, and also to the following report about the thematic discussions.

	number of papers and posters in				total number per category
	CERME-6	CERME-7	CERME-8	CERME-9	
1 theoretical frameworks for including history	1.33 (P) .5	2.25		2.1 (P) .2	5.68 (P) .7
2 relation between history and other parts of MathEd	1.83	.5	.25 (P) .45	.25	2.83 (P) .45
3 role of history at primary, secondary, tertiary level	not yet present	3 (P) .75	3.05 (P) .2	.8 (P) .5	6.85 (P) 1.45
4 role of history in teacher Ed	1.17	2	.75 (P) .7	2.2 (P) .8	6.12 (P) 1.5
5 parallelism of historical & cognitive development	(P) .5		.5 (P) .75	1.05	1.55 (P) 1.25
6 integrating original sources in Math classes	1.67	.75 (P) .5	1.8	.3 (P) .5	4.52 (P) 1
7 history in curricula, textbooks, classrooms	1.33	(P) .5	.25	.3	1.88 (P) .5
8. design & assessment of Ed materials	1.33	.5 (P) .25	2.4	1.3	5.53 (P) .25
9a history OF Math Ed general	1	.75 (P) .5	.5	2.2	4.45 (P) .5
9b history of actual teaching practice		.25 (P) .25	.25	.4	0.9 (P) .25
9c history of teaching and learning materials	1.67	.75 (P) .75	.5 (P) .5	2.2	5.12 (P) 1.25
9d history of MathEd from a social/cultural viewpoint	1.67	2.25 (P) 1.5	1.75 (P) .4	.9	6.57 (P) 1.9
	13 1 poster	13 5 posters	12 3 posters	14 2 posters	52 11 posters

Table 1. Amount of pages in papers and posters concerning different themes in the history WG.

Development of the thematic discussions

In the following we describe how the sessions of the working group were organized and in particular we focus on the thematic discussions regarding more general themes related to and exemplified by the papers and posters presented in the group. We also briefly report the outcome of these thematics discussions.

CERME-6

At CERME-6 a rather traditional structure was chosen for the WG. The time for each paper was equally divided between presentation and discussion. Some of the general discussion points after the presentations included: “research methodology, historical references, educational and mathematical points” (Furinghetti et al. 2010, p. 2681). Another crucial aspect following the first conduction of the WG was the realization of the relevance of history *of* mathematics education for current teaching practice.

CERME-7

Based on the sessions in CERME-6 the co-chairs felt the need for further reflection upon aspects of methodology and upon connections with mathematics education research more in general. For these reasons, four general themes were identified and discussed in special sessions:

1. Research questions and relevance of the research
2. Use of HPM theoretical constructs vs. mathematics education theoretical constructs
3. Methods, data and analysis
4. Validity, reliability and generality of the reported research results

Some time for presentations was shifted to group discussions of the general themes. Due to a relatively high number of young researchers in the group, the co-chairs decided to have some sessions in which the young researchers in one group discussed theme 1 under the guidance of an experienced researcher, while the others discussed theme 2 in another group. It led several of the young “participants to reconsider their research aim(s), formulate questions, refine formulations of existing questions, or expand their research perspectives. Also, the discussion of theory-driven versus problem-driven research led to discussions of the role of theory in (empirical) research, etc.” (Jankvist et al., 2011, p. 1638). In the other group the following key-issues were identified as “crucial for the domain of history in mathematics education: *[… to] provide some order in the wide spectrum of research and implementations […], to somehow check the efficiency of introducing a historical dimension, […] to convince the target population [...]; and to develop appropriate conditions for designing, realizing, and evaluating our research*” (ibid., p. 1638, italics in original). For themes 3 and 4, new discussion groups were formed, now researchers in history *of* mathematics education formed one subgroup and researchers in history *in* mathematics education another. These discussions made the different methodological

approaches of the WG's two strands even more evident, but at the same time also made clear some of the overlaps.

Finally, it was decided that in the future an effort should be made to make it easier for the poster-presenters, who oftentimes are the young researchers, to join in in the thematics discussions by providing them with short timeslots for briefly and early presenting their posters.

CERME-8

For CERME-8 five themes for thematic discussions were identified based on the submitted papers:

1. Interdisciplinarity
2. Theoretical frameworks for history *of* mathematics education
3. History in high school/upper secondary school mathematics education
4. History in pre-high school mathematics education
5. History of mathematics in teacher education

New in CERME-8 was that discussion questions were ready and sent out prior to the congress. We exemplify here the thematic discussions of themes 2 and 3.

For theme 2 the questions asked were: What is in this respect the difference between 'story' and 'history'? A supplemental suggestion was made that the difference might very well be related to the use of various (theoretical) frameworks. The next questions addressed which theoretical frameworks are actually available for research in this field? And a third question raised the issue of to what extent the history of mathematics education requires study of primary or original sources, documents, etc.? In the final report, the following is written:

> [T]here was a consensus about *story* being something narrative, whereas *history*, although it may contain narratives (or stories), is structured by theoretical frameworks, the purpose of which includes being able to see benefits or limitations, to communicate results, and to enable the researchers to organize and present findings, assertions, etc. [...] The participants point to for example constructs from history research, e.g., those of more externalistic historiography of studying factors crucial to the development of institutions, etc. [...] As to the role of primary sources, all consider these practically a necessity for conducting history of mathematics education." (Jankvist et al. 2013, pp. 1947-1948)

Questions for theme 3 were: What are the special challenges when using history in primary school, kindergarten, etc.? How do we stay 'true' to history, i.e. non-Whig, when applying history of mathematics at pre high school levels? How do we determine the effect of history, as opposed to the use of physical materials or other interventions (e.g., drama)? Some conclusions were:

> ... in practice when using history at younger age levels there is a need for compromise, also in order to make the mathematics itself more accessible to the children [...] there may be the need for narratives in the form of telling stories of mathematics, rather than confronting them with the actual history of mathematics. But as one of the subgroups state in their report: "You have to tell stories, but the knowledge of history enables you to tell *true* stories." (ibid., p. 1948)

CERME-9

This time around, four overarching themes for the thematic discussions were identified:

1. Meta-level or methodological reflections
2. History *of* mathematics education
3. History *in* mathematics education – student perspective
4. History *in* mathematics education – teacher perspective

We shall exemplify themes 1 and 4. Discussions during this WG were carried out both in small-groups and in plenum.

Theme 1 centered around the questions: What (if any) is (could be) the role assigned to epistemological/historical reflection in some major mathematics education theoretical frameworks: e.g. TDS; ATD; APOS; MKT; etc. In regard to the local/global tension: Can large-scale surveys (e.g. history of algebra, notion of proof from Euclid to Hilbert, evolution of the concept of function, etc.) go beyond the 'bird's eye view'? Can we elicit necessary conditions for such large-scale surveys to make any sense? As for the outcome of discussions related to this theme, "several participants shared the view that using a general survey of history (i.e., "global view") helps to create a cultural landscape, which includes and accommodates multiple tools, concepts, and ideas – and which establishes a meaningful lens to use from the outset" (Jankvist et al. 2016, pp. 1780-1781).

As for theme 4 the main question was: What minimal/satisfactory level of command of history of mathematics can we reasonably attempt to achieve in teacher training? During discussions, this question was addressed through two 'sub-issues': criteria for being an *able reader* (such as having the ability to assess a primary source with a critical mind) and *the epistemological tool-box* (which contains the descriptive/analytical concepts we wish to provide to teachers). Finally, it was discussed to what extent we shall expose (future)-teachers to elements of history of mathematics which have no direct connections with classroom contents (in particular to enrich their "image" of the parts of higher mathematics, which they studied but will not teach). Reflections, also methodological ones, on actual teacher-training modules were made. The discussions also led to further questions regarding challenges, e.g. how to go beyond the mere 'raise awareness' objective, how to objectify the impact on student-teachers, and not least how to stabilize any potential impact?

The `twee' Cs of ERME

One characteristic of ERME is its three Cs. We recall former ERME president Ferdinando Azzarello, usually while wearing a bowtie, reciting *Communication*, *Co-operation* and *Collaboration* - stressing their importance for ERME's activities. Consulting *Oxford English Dictionary* (*OED*) we find it almost impossible to distinguish between Co-operation and Collaboration. Collaboration has a second, pejorative interpretation: "Traitorous cooperation with the enemy", while co-operation is always positive. *OED* gives as the first meaning of collaboration: "United labour, co-operation; esp. in literary, artistic, or scientific work.". That is what ERME promotes. A co-operation can also be a "combination of a number of persons" who work together for their common benefit. Co-operation is cited by *OED* as one of the meanings of collaboration, and not the other way round, Or should we stay with "United labour", which points at heavy work? That would connect well with mathematics education. We are confused, and this is only reinforced by the translations of both words into Danish ('samarbejde') and Dutch ('samenwerking'). In these languages the words for united labour and co-operation are the same. The case is different for the adjectives co-operative and collaborative (the first indicates a positive intention, the second is more about organisation; a collaborative book does not presuppose co-operative authors). We use co-operation and collaboration as two interchangeable concepts. And for this section's title we coin the numeral 'twee', which can be two or three. Twee (pronounced ' tway') happens also to be Dutch for 2.

Much of the **C**ommunication the history WD took place during the thematic discussions, which to a large degree have assisted in forming an 'identity' for the participants of the group and over time has led to some 'common standards' on how to conduct research in the field of the WG. An example is paper 4 by Alpaslan, Schubring and Günergun, where frameworks developed by a German researcher (Schubring, 2000) and a Danish researcher (Niss, 1996) inform a cultural and historical case from the Ottoman empire. Furthermore this is an example of general mathematics education informing not history *in* but history *of* mathematics education. Paper 4 is also a fair illustration of **C**ollaboration between a young researcher and two experienced researchers. In fact, Alpaslan had participated in the CERME history group since CERME-7 when being a master's student. At both CERME-7 and CERME-8 he presented papers related to history *in* mathematics education. No doubt, due to his participation in the CERME group, Alpaslan had become acquainted with the history *of* mathematics education, so he began researching the history of mathematics education in Turkey, which led to the joint paper 4 with experienced scholars.

Related to **C**ommunication, in order to get the more established mathematics education community to pay attention to the potential benefits of including history in the teaching and learning of mathematics, the 'history people' need to 'speak' the established language of this community. In this sense, the CERME group may be seen as a spearhead in getting the HPM 'hinterland', which has its own idiom and frameworks, to cooperate towards being understood by the education research community. Also, the CERME working group has provided a perfect

habitat for testing out which general mathematics education frameworks, notions, etc. would apply to history and mathematics education. Paper 2 is a good example of this. It reports about classroom experiments about the building of the concept of tangent. The author investigated if studying and discussing historical appearances of this concept would help students in their concept building. Paper 2 is also interesting from the perspective of **C**ommunication, since CERME gave the author, who was preparing a PhD the possibility to present and discuss her work with several experts in her field. This also holds for paper 3, which has other valuable aspects as well. CERME gave the author, who is from Brazil and who was working within the French research culture, the possibility to communicate with researchers from other research cultures.

The CERME group on history reflects the diversity of the HPM community, by having participants who are mainly researchers of the history of mathematics or mathematics education, educators with an elaborate interest in history of mathematics and its inclusion *in* mathematics education or the history *of* mathematics education, or some combination of these. In hindsight this diversity, as illustrated by Table 1 and the thematic discussions, has fostered rich discussions and reflections. An example is Paper 1 by Kjeldsen, who inspired by the Danish historian Jensen (2010), proposed to distinguish between a pragmatic and a scholarly approach to history, and between observer history and actor history. At the time of CERME-7, this fostered intense discussion among the more experienced researchers who asked from which other fields history in mathematics education gets it inspiration in terms of frameworks, theoretical constructs, etc. The conclusion was that it does so from a wide range of fields (from mathematics to the social sciences, etc.). More interestingly this discussion led to another reflection, namely that in order to 'do well' in relation to research on history *in* or *of* mathematics education, one must possess a profound knowledge of the history of mathematics - and/or the history of mathematics education - which indeed requires a firm mathematical background, and at the same time one must be well read in the mathematics education literature. Now, since this 'trinity' (history, mathematics, and education) of knowledge is not always easy to come by, the area of history and mathematics education is in fact a perfect place for **C**ollaboration between researchers with different backgrounds and areas of knowledge.

Summarizing, CERME is important as a platform where young and established researchers are open to meet each other, and also as a platform where research cultures meet. Next, the rules how to behave on this platform, especially the requirement to submit a paper, to peer-review papers and to read the papers before the conference, give CERME and its participants a fair chance to be successful. Table 1 shows that this has led to a considerable and diverse output. For us, this indicates once again the value of historical knowledge and research for mathematics education.

References

Alpaslan, M., Schubring, G. & Günergun, F. (2015), 'Mebahis-i İlmiye' as the first periodical on mathematical sciences in the Ottoman Turkey. In N. Vondrová and K. Krainer (Eds.):

Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education, pp. 1783-1789. Prague: Charles University.

da Costa, D.A. (2009), Arithmetic in primary school in Brazil: end of the nineteenth century. In V. Durand-Guerrier, S. Soury-Lavergne and F. Arzarello (Eds.): *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*, pp. 2712-2721. Institut national de recherche pédagogique.

Fasanelli, F. & Fauvel, J. (2006). History of the International Study Group on the Relations Between the History and Pedagogy of Mathematics: the first twenty-five years, 1976-2000, pp. x-xxviii in: Furinghetti, F., Kaijser, S., & Tzanakis, C. (eds.), *Proceedings of HPM 2004 & ESU 4: ICME 10 Satellite Meeting of the HPM Group & Fourth European Summer University on the History and Epistemology in Mathematics Education,* Iraklion, Greece: University of Crete

Fauvel, J. (1990). *History in the Mathematics Classroom. The IREM Papers.* Leicester: The Mathematical Association.

Fauvel, J. (1991). Using history in mathematics education, *For the learning of mathematics 11*(2)*,* 3-6.

Fauvel, J. & van Maanen, J. (Eds.) (2000). *History in Mathematics Education. The ICMI Study*. Dordrecht: Kluwer Academic Publishers.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*(4), 391-408.

Fried, M.N. & Jahnke, H. N. (2015). Otto Toeplitz's 1927 Paper on the Genetic Method in the *Teaching* of Mathematics. *Science in Context 28*, 285-295 doi:10.1017/ S0269889715000034

Furinghetti, F., Dorier, J. L., Jankvist, U. T., van Maanen, J. & Tzanakis, C. (2010). Introduction - theory and research on the role of history in mathematics education. In V. Durand-Guerrier, S. Soury-Lavergne and F. Arzarello (Eds.): *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*, pp. 2679-2681. Institut national de recherche pédagogique.

Jankvist, U. T. (2012). A first attempt to identify and classify empirical studies on 'History in Mathematics Education'. Chapter 13 in Sriraman, B. (Ed.) *Crossroads in the History of Mathematics and Mathematics Education*. Pages 295-332. The Montana Mathematics Enthusiast Monographs 12. Information Age Publishing..

Jankvist, U. T., Mosvold, R., Jakobsen, A. & Fauskanger, J. (2015). Analysing the use of history of mathematics through MKT. *International Journal of Mathematical Education in Science and Technology*, *46*(4), 495-507.

Jankvist, U. T., Lawrence, S., Tzanakis, C. & van Maanen, J. (2011). Introductory report from WG12: History in mathematics education. In M. Pytlak, T. Rowland and E. Swoboda (Eds.): *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education,* pp. 1636-1639. Rzeszów: University of Rzeszów.

Jankvist, U. T., Clark, K., Lawrence, S. & van Maanen, J. (2013). Introduction to the papers and posters of WG12: History in mathematics education. In B. Ubuz, C. Haser and M. A. Mariotti (Eds.): *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education*, pp. 1945-1950. Ankara: METU.

Jankvist, U. T., Chorlay, R., Clark, K. & Lawrence, S. (2016). Introduction to the papers and posters of TWG12: History in mathematics education. In N. Vondrová and K. Krainer (Eds.): *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education,* pp. 1779-1781. Prague: Charles University.

Karp, A. & Schubring, G. (eds.) (2014). *Handbook on the History of Mathematics Education*. New York, NY: Springer.

Katz, V. J. (Ed.) (2000). *Using history to teach mathematics - an international perspective*. Washington, DC: MAA, the Mathematical Association of America.

Katz, V. J. & Tzanakis, C. (Eds.) (2011). *Recent developments on introducing a historical dimension in mathematics education*. Washington, DC: MAA, the Mathematical Association of America.

Katz, V. J., Jankvist, U. T., Fried, M. N. & Rowlands, S. (2014). Special issue on history and philosophy of mathematics in mathematics education. *Science & Education*, *23*(1).

Kjeldsen, T. H. (2011). Uses of history in mathematics education: development of learning strategies and historical awareness. In M. Pytlak, T. Rowland and E. Swoboda (Eds.): *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education,* pp. 1640-1649. Rzeszów: University of Rzeszów.

NCTM Yearbook (1969). *Historical topics for the mathematics classroom*. Reston, Virginia: NCTM. (Updated edition 1989).

Matthews, M. R. (Ed.) (2014). *International Handbook of Research in History, Philosophy and Science Teaching*. Dordrecht: Springer.

Mota, C., Ralda, M. E. & Estrada, M. F. (2013). The teaching of the concept of tangent line using original sources. In B. Ubuz, C. Haser and M. A. Mariotti (Eds.): *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education*, pp. 2048-2057. Ankara: METU.

Niss, M. (1996). Goals of mathematics teaching. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), International handbook of mathematics education (pp. 11–48, Chapter 1). Dordrecht, The Netherlands: Kluwer Academic.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning ideas and inspiration for the development of mathematics teaching and learning in Denmark*. English Edition, October 2011. IMFUFA tekst no. 485. Roskilde: Roskilde University. [Published in Danish in 2002].

Popp, W. (1968). *Geschichte der Mathematik im Unterricht*, München: Bayerische Schulbuch Verlag; translated into English as *History of mathematics: topics for schools,* London: Transworld Publishers (1975) and Milton Keynes: The Open University Press (1978)

Radford, L., Furinghetti, F. & Katz, V. J. (Eds.) (2007). The history of mathematics: theory and practice. Special issue in *Educational Studies in Mathematics*, *66*(2).

Schubring, G. (2000). The first international curricular reform in mathematics and the role of Germany – A case study in the transmission of concepts. In A. Gagatsis, C. P. Constantinou, & L. Kyriakides (Eds.), Learning and assessment in mathematics and science (pp. 265–287). Nicosia, Cyprus: Department of Education, University of Cyprus.

Toeplitz, O. (1927). Das Problem der Universitätsvorlesungen über Infinitesimalrechnung und ihrer Abgrenzung gegenüber der Infinitesimalrechnung an den höheren Schulen. *Jahresbericht der deutschen Mathematiker-Vereinigung, XXXVI*, 88–100.